

# Trend to Equilibrium for the Becker-Döring Equations: An Analogue of Cercignani's Conjecture

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**ABSTRACT.** In this work we investigate the rate of convergence to equilibrium for sub-critical solutions to the Becker-Döring equations with physically relevant coagulation and fragmentation coefficients and mild assumptions on the given initial data. Using a discrete version of the log-Sobolev inequality with weights we show that in the case where the coagulation coefficient grows linearly and the detailed balance coefficients are of typical form, one can obtain a linear functional inequality for the dissipation of the relative free energy. This results in showing Cercignani's conjecture for the Becker-Döring equations and consequently in an exponential rate of convergence to equilibrium. We also show that for all other typical cases one can obtain an 'almost' Cercignani's conjecture that results in an algebraic rate of convergence to equilibrium. Additionally, we show that if one assumes an exponential moment condition one can recover Jabin and Niethammer's rate of decay to equilibrium, i.e. an exponential to some fractional power of  $t$ .

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## 1. INTRODUCTION

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**1.1. The Becker-Döring Equations.** The Becker-Döring equations are a fundamental set of equations which describe the kinetics of a first order phase transition. Amongst the phenomena to which it applies one can find crystallisation, vapour condensation, aggregation of lipids and phase separations in alloys (see for instance [27, 22]).

The Becker-Döring equations give the time evolution of the size distribution of clusters of a certain substance. Denoting by  $\{c_i(t)\}_{i \in \mathbb{N}}$ , the density of clusters of size  $i$  at time  $t \geq 0$  (i.e. the density of clusters that are composed of  $i$  particles), the equations read

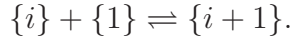
$$(1.1a) \quad \frac{d}{dt}c_i(t) = W_{i-1}(t) - W_i(t), \quad i \in \mathbb{N} \setminus \{1\},$$

$$(1.1b) \quad \frac{d}{dt}c_1(t) = -W_1(t) - \sum_{k=1}^{\infty} W_k(t),$$

where

$$(1.2) \quad W_i(t) := a_i c_1(t) c_i(t) - b_{i+1} c_{i+1}(t) \quad i \in \mathbb{N}.$$

and  $a_i, b_i$ , assumed to be strictly positive, are the *coagulation and fragmentation coefficients*. They determine the rate at which clusters of size  $i$  combine with clusters of size 1 to create a cluster of size  $i + 1$ , or clusters of size  $i + 1$  break into clusters of size  $i$  and 1. This corresponds to the basic assumption of the underlying model: if we represent symbolically by  $\{i\}$  the species of clusters of size  $i$ , then the only (relevant) chemical reactions that take place are



The quantity  $W_i(t)$  defined in (1.2) represents the *net rate* of the reaction  $\{i\} + \{1\} \rightleftharpoons \{i + 1\}$ , and under the above set of equations it is easy to see that the *density*, or *mass*, of the solution, defined by

$$(1.3) \quad \varrho := \sum_{i=1}^{\infty} i c_i(0) = \sum_{i=1}^{\infty} i c_i(t)$$

is conserved under the Becker-Döring evolution. The original equations proposed by Becker and Döring [5] were similar to (1.1), with the slight change that the density of one particle  $c_1$ , usually called the monomer density, was assumed to be constant. The current version, motivated by the conservation of total density, was first discussed in [7] and [25] and is widely used in classical nucleation theory. For more information about the physical background and applications of the equations we refer the interested reader to the aforementioned works as well as the recent reviews [27, 22].

Much like in other kinetic equations, the study of a state of equilibrium and the convergence to it is a fundamental question in the study of the Becker-Döring equations. Defining the *detailed balance coefficients*  $Q_i$  recursively by

$$(1.4) \quad Q_1 = 1, \quad Q_{i+1} = \frac{a_i}{b_{i+1}} Q_i \quad i \in \mathbb{N}$$

one can see that for a given  $z \geq 0$  the sequence

$$(1.5) \quad c_i = Q_i z^i$$

is formally an equilibrium of (1.1). However, depending on the coagulation and fragmentation coefficients  $a_i$  and  $b_i$ , many of these formal equilibria do not have a finite mass. The largest  $z_s \geq 0$ , possibly  $z_s = +\infty$ , for which

$$\sum_{i=1}^{\infty} i Q_i z^i < +\infty \quad \text{for all } 0 \leq z < z_s$$

is called the *critical monomer density*, or sometimes the monomer saturation density. The *critical mass* (or, again, saturation mass) is then defined by

$$(1.6) \quad \varrho_s := \sum_{i=1}^{\infty} i Q_i z_s^i \in [0, +\infty].$$

It is important to note that both  $z_s$  and  $\varrho_s$  are uniquely determined by  $a_i$  and  $b_i$  and that  $\{Q_i z^i\}_{i \in \mathbb{N}}$  is a finite-mass equilibrium only for  $0 \leq z < z_s$ , with the possibility for the equality  $z = z_s$  only when  $\varrho_s < +\infty$ . Additionally, it is easy to see that for a given finite mass  $\varrho \leq \varrho_s$  there exists a unique  $\bar{z} \geq 0$  such that

$$\varrho = \sum_{i=1}^{\infty} i Q_i \bar{z}^i,$$

giving us a candidate for the asymptotic equilibrium state of (1.1) under a given initial data. These are in fact the only finite-mass equilibria (see [3]), and  $\bar{z}$  defined above is called the *equilibrium monomer density* for a given mass  $\varrho$ .

A finite mass equilibrium is called *subcritical* when its mass  $\varrho$ , is strictly less than  $\varrho_s$ . It is called *critical* if  $\varrho = \varrho_s$  and *supercritical* if  $\varrho > \varrho_s$ , assuming  $\varrho_s < +\infty$ . In this paper we will only concern ourselves with subcritical solutions. Thus, to avoid triviality we always assume that  $z_s > 0$ .

The critical density  $\varrho_s$ , if finite, marks a change in the behaviour of equilibrium states: if  $\varrho < \varrho_s$  then a unique equilibrium state with mass  $\varrho$  exists, while if  $\varrho > \varrho_s$  no such equilibrium can occur and a phase transition phenomenon takes place — reflected in the fact that the excess density  $\varrho - \varrho_s$  is concentrated in larger and larger clusters as time progresses.

**1.2. Typical Coefficients.** Physically motivated coagulation and fragmentation coefficients are often given by

$$(1.7) \quad a_i = i^\gamma, \quad b_i = a_i \left( z_s + \frac{q}{i^{1-\mu}} \right), \quad i \in \mathbb{N},$$

for some  $0 < \gamma \leq 1$ ,  $z_s > 0$ ,  $q > 0$  and  $0 < \mu < 1$  (see [23, 21] for details and concrete examples).

A different kind of reasoning, based on a statistical mechanics argument involving the binding energy of clusters, results in the coefficients

$$(1.8) \quad a_i = i^\gamma, \quad b_i = z_s(i-1)^\gamma \exp(\sigma i^\mu - \sigma(i-1)^\mu), \quad i \in \mathbb{N},$$

for appropriate constants  $\gamma, \mu$  and  $\sigma$  (see for instance [10, 16, 19, 20]). The behaviour of (1.7) and (1.8) is similar: for both of them we can write (by definition of  $Q_i$ )

$$(1.9) \quad Q_i = \frac{a_1 a_2 \dots a_{i-1}}{b_2 b_3 \dots b_i} = z_s^{1-i} \alpha_i,$$

where  $\{\alpha_i\}_{i \in \mathbb{N}}$  is non-increasing and satisfies

$$\lim_{i \rightarrow \infty} \frac{\alpha_{i+1}}{\alpha_i} = 1.$$

Our results are valid for both types of coefficients (1.8) and (1.7), which are often used in the literature and cover a large range of applicable cases.

**1.3. Previous Results.** Let us briefly review existing results on the mathematical theory of the Becker-Döring equations, which has advanced much since the first rigorous works on the topic [3, 1]. In [3] the authors showed existence and uniqueness of a global solution to (1.1) when

$$(1.10) \quad a_i \leq C_1 i, \quad b_i \leq C_2 i, \quad \sum_{i=1}^{\infty} i^2 c_i(0) < +\infty,$$

for some constants  $C_1, C_2 > 0$ . As expected, under the above assumptions the unique solution conserves mass (this is, (1.3) holds rigorously). This basic existence theory is applicable to all solutions we consider in this work.

The asymptotic behaviour of solutions to (1.1) is one of the most interesting aspects of the equation. Supercritical behaviour, while not dealt with in this work, has a particularly interesting link to late-stage coarsening and has been studied extensively in [24, 28, 13, 21]. Asymptotic approximations of such solutions have been developed in [19, 15, 16].

Regarding the subcritical regime, it was proved in [3, 1] that solutions with subcritical mass  $\varrho$  approach the unique equilibrium with this mass (determined by (1.3)). A fundamental quantity in understanding this approach is the *free energy*,  $H(\mathbf{c})$ , defined (at least formally) for any sequence  $\mathbf{c} = \{c_i\}_{i \in \mathbb{N}}$  by

$$(1.11) \quad H(\mathbf{c}) := \sum_{i=1}^{\infty} c_i \left( \log \frac{c_i}{Q_i} - 1 \right).$$

It can be shown that  $H(\mathbf{c}(t))$  decreases along solutions  $\mathbf{c} = \mathbf{c}(t)$  to the Becker-Döring equations; in fact, for a (strictly positive, suitably decaying for large  $i$ ) solution  $\mathbf{c}(t) = \{c_i(t)\}_{i \in \mathbb{N}}$  of (1.1) we have

$$(1.12) \quad \begin{aligned} \frac{d}{dt} H(\mathbf{c}(t)) &= -D(\mathbf{c}(t)) \\ &:= - \sum_{i=1}^{\infty} a_i Q_i \left( \frac{c_1 c_i}{Q_i} - \frac{c_{i+1}}{Q_{i+1}} \right) \left( \log \frac{c_1 c_i}{Q_i} - \log \frac{c_{i+1}}{Q_{i+1}} \right) \leq 0. \end{aligned}$$

This free energy is motivated by physical considerations and constitutes a Lyapunov functional for our equation. Since it does not have a definite sign we define a more natural candidate to measure the distance of  $\mathbf{c}(t) = \{c_i(t)\}_{i \in \mathbb{N}}$  to the equilibrium. Using the notation

$$(\mathcal{Q}_z)_i = Q_i z^i$$

and denoting by  $\mathcal{Q} = \mathcal{Q}_{\bar{z}}$ , we can define the *relative free energy* as

$$(1.13) \quad H(\mathbf{c}|\mathcal{Q}) := \sum_{i=1}^{\infty} c_i \left( \log \frac{c_i}{\bar{z}^i Q_i} - 1 \right) + \sum_{i=1}^{\infty} \bar{z}^i Q_i = H(\mathbf{c}) - \log \bar{z} \sum_{i=1}^{\infty} i c_i + \sum_{i=1}^{\infty} \bar{z}^i Q_i.$$

The relative free energy has the same time derivative as the free energy, and thus the same monotonicity property

$$\frac{d}{dt} H(\mathbf{c}(t)|\mathcal{Q}) = -D(\mathbf{c}(t)) \quad \forall t \geq 0,$$

where the *free energy dissipation*  $D$  is defined in (1.12). The relative free energy also satisfies

- $H(\mathbf{c}|\mathcal{Q}) \geq 0$ , as can be seen by writing

$$(1.14) \quad H(\mathbf{c}|\mathcal{Q}) = \sum_{i=1}^{\infty} \mathcal{Q}_i \varphi\left(\frac{c_i}{\mathcal{Q}_i}\right), \quad \text{with } \varphi(r) := r \log r - r + 1 \geq 0$$

- $H(\mathbf{c}|\mathcal{Q}) = 0$  if and only if  $c_i = \mathcal{Q}_i = \mathcal{Q}_i \bar{z}^i$  for any  $i \in \mathbb{N}$ , which is readily seen from (1.14).

This hints that  $H(\mathbf{c}|\mathcal{Q})$  is the right distance to investigate. Indeed, while strictly speaking  $H(\mathbf{c}|\mathcal{Q})$  is not a distance, it does control the  $\ell^1$  distance between  $\mathbf{c}$  and  $\mathcal{Q}$  by the celebrated Csiszár-Kullback inequality<sup>1</sup>, which in our case translates to

$$(1.15) \quad \|\mathbf{c} - \mathcal{Q}\|_{\ell^1(\mathbb{N})} = \sum_{i=1}^{\infty} |c_i - \mathcal{Q}_i| \leq \sqrt{2\rho H(\mathbf{c}|\mathcal{Q})}.$$

The issue of estimating the rate of convergence to equilibrium of subcritical solutions is the main concern of this paper. The first result in this direction was the work [17] by Jabin and Niethammer, where they investigated the possibility of applying the so-called *entropy method* to the Becker-Döring equation. This consists roughly in looking for functional inequalities between a suitable Lyapunov functional of the equation (generally called the entropy; it corresponds to the relative free energy in our case) and its dissipation, so that one obtains a differential inequality that estimates the rate of convergence to equilibrium. In the case of the Becker-Döring equation, it was proved in [17] that there exists a constant  $C > 0$ , depending only on the fixed parameters of the problem and the initial data, such that

$$(1.16) \quad D(\mathbf{c}) \geq C \frac{H(\mathbf{c}|\mathcal{Q})}{(\log H(\mathbf{c}|\mathcal{Q}))^2},$$

for all nonnegative sequences  $\mathbf{c}$  with subcritical mass  $\rho$ , satisfying  $\epsilon \leq c_1 \leq z_s - \epsilon$  for some  $\epsilon > 0$  and

$$\sum_{i=1}^{\infty} e^{\mu_i} c_i = M^{\text{exp}} < +\infty.$$

The constant  $C$  depends on  $\epsilon$  and  $M^{\text{exp}}$ . This result applies under reasonable conditions on the coefficients  $a_i$  and  $b_i$ ; in particular, it applies to the coefficients (1.7) and (1.8). If we consider now a solution  $\mathbf{c} = \mathbf{c}(t)$  to (1.1), we may apply the inequality (1.16) to  $\mathbf{c}(t)$  as long as  $\mathbf{c}(t)$  satisfies the appropriate conditions, obtaining

$$\frac{d}{dt} H(\mathbf{c}(t)|\mathcal{Q}) = -D(\mathbf{c}(t)) \leq -C \frac{H(\mathbf{c}(t)|\mathcal{Q})}{(\log H(\mathbf{c}(t)|\mathcal{Q}))^2}.$$

Adding to this some additional considerations for the times  $t$  for which the inequality (1.16) is not applicable to  $\mathbf{c}(t)$ , one can deduce that  $H(\mathbf{c}(t)|\mathcal{Q})$  is (essentially) bounded above by the solution of the above differential inequality, namely that

$$H(\mathbf{c}(t)|\mathcal{Q}) \leq H(\mathbf{c}(0)|\mathcal{Q}) e^{-Kt^{\frac{1}{3}}}$$

for some  $K > 0$ . Using inequality (1.15), this gives an almost-exponential rate of convergence to equilibrium for subcritical solutions in the  $\ell^1(\mathbb{N})$  norm.

The question remained open of whether the convergence is in fact exponential or not. Recently this has been answered positively by two of the authors [10] through a different approach involving a detailed study of the spectrum of the linearisation of equation (1.1) around a subcritical equilibrium. This is an approach with a strong analogy to results

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<sup>1</sup>Sometimes called Pinsker or Kullback-Pinsker inequality.

in the theory of the Boltzmann equation; we refer to [29, 10] for more details on this parallel. The idea of the argument is to use the inequality (1.16) when one is far from equilibrium. Then, once we have reached a region which is close enough to equilibrium, the linearised regime is dominant and one can use the spectral study of the linearised operator in order to show that the convergence is in fact exponential. The outcome of this strategy is the following: for many interesting coefficients (including (1.7) and (1.8)), subcritical solutions  $\mathbf{c} = \mathbf{c}(t)$  to (1.1) with

$$\sum_{i=1}^{\infty} e^{\mu i} c_i(0) := M^{\text{exp}} < +\infty \quad \text{for some } \mu > 0$$

satisfy that

$$\sum_{i=1}^{\infty} e^{\mu' i} |c_i(t) - Q_i| \leq C e^{-\lambda t} \quad \text{for } t \geq 0$$

for some  $0 < \mu' < \mu$ ,  $C > 0$  and  $\lambda > 0$  which depend on the parameters of the problem and on  $M^{\text{exp}}$ . In fact,  $\mu$  and  $C$  only depend on the initial data  $\mathbf{c}(0)$  through its mass and the value of  $M^{\text{exp}}$ ;  $\lambda$  depends only on the coefficients and the initial mass and can be estimated explicitly. The value of  $\lambda$  is bounded above by (and can be taken very close to) the size of the spectral gap of the linearised operator. Recently Murray and Pego [18] have used this spectral gap and developed the local estimates of the linearised operator in order to obtain convergence to equilibrium at a polynomial rate with milder conditions on the decay of the initial data. These results, like those in [10], are local in nature and require the use of some global estimate such as (1.16) in order to provide global rates of convergence to equilibrium.

**1.4. Main Results.** Our main goal in this work is to complete the picture of convergence to equilibrium by investigating modified and improved versions of the inequality (1.16). We show optimal inequalities and settle the question of whether full exponential convergence can be obtained through a *linear* inequality of the form

$$D(\mathbf{c}) \geq K H(\mathbf{c}|\mathcal{Q}).$$

in some cases. In analogy to the Boltzmann equation, we refer to the question of whether such  $K$  exists along solutions to (1.1) as *Cercignani's conjecture for the Becker-Döring equations*. In fact, we show that under relatively mild conditions on the initial data, typical coagulation and fragmentation coefficients (such as (1.7) and (1.8)) admit an “almost” Cercignani conjecture for the energy dissipation, i.e. an inequality bounding below  $D(\mathbf{c})$  by a power of  $H(\mathbf{c}|\mathcal{Q})$ , yielding an explicit rate of convergence to equilibrium. Surprisingly, we also find a relevant case ( $a_i \sim i$  for all  $i$ ) for which the conjecture is actually valid.

We will often require the following assumptions on the coagulation and fragmentation coefficients. Some of these are similar to those in [17], and always include coefficients of the form (1.7) and (1.8). We recall that we always assume  $a_i, b_i > 0$  for all  $i \in \mathbb{N}$ , and that the detailed balance coefficients  $Q_i$  were defined in (1.4) — given  $a_i$  one can determine  $b_i$  through  $Q_i$ , and vice versa.

**Hypothesis 1.**  $0 < z_s < +\infty$ .

**Hypothesis 2.** For all  $i \in \mathbb{N}$ ,  $Q_i = z_s^{1-i} \alpha_i$ , where  $\{\alpha_i\}_{i \in \mathbb{N}}$  is a non-increasing positive sequence,  $\alpha_1 > 0$  and  $\lim_{i \rightarrow \infty} \frac{\alpha_{i+1}}{\alpha_i} = 1$ .

**Hypothesis 3.**  $a_i = O(i^\gamma)$  for some  $0 < \gamma \leq 1$ , i.e. there exist  $C_1, C_2 > 0$  such that

$$C_1 i^\gamma \leq a_i \leq C_2 i^\gamma \quad \text{for all } i \in \mathbb{N}.$$

In most of the estimates we obtain, a crucial role will be played by the *lower free energy dissipation*,  $\overline{D}(\mathbf{c})$ , defined for a given non-negative sequence  $\mathbf{c}$  by

$$(1.17) \quad \overline{D}(\mathbf{c}) = \sum_{i=1}^{\infty} a_i Q_i \left( \sqrt{\frac{c_1 c_i}{Q_i}} - \sqrt{\frac{c_{i+1}}{Q_{i+1}}} \right)^2$$

At this point one notices that the elementary inequality  $(x-y)(\log x - \log y) \geq (\sqrt{x} - \sqrt{y})^2$  when  $x, y > 0$  implies that

$$D(\mathbf{c}) \geq \overline{D}(\mathbf{c})$$

for any non-negative sequence  $\mathbf{c}$ . Thus, any lower bound that is obtained for  $\overline{D}(\mathbf{c})$  will transfer immediately to  $D(\mathbf{c})$ .

We now state our main result on general functional inequalities for the free energy dissipation, from which later we conclude a quantitative rate of convergence to equilibrium. It can be divided in two parts: functional inequalities when  $c_1$  is not too small, nor is too far from  $\overline{z}$ , and inequalities in the case where  $c_1$  escapes the above region.

**Theorem 1.1.** *Let  $\{a_i\}_{i \in \mathbb{N}}, \{Q_i\}_{i \in \mathbb{N}}$  satisfy Hypotheses 1-3 with  $0 \leq \gamma \leq 1$  and let  $\mathbf{c} = \{c_i\}_{i \in \mathbb{N}}$  be an arbitrary positive sequence with finite total density  $0 < \varrho < \varrho_s$ .*

(i) **(Estimate for  $a_i \sim i$ .)** *Assume that  $\gamma = 1$  and that there exist  $\delta > 0$  such that*

$$(1.18) \quad \delta < c_1 < z_s - \delta.$$

*Then there exists  $K > 0$  depending only on  $\delta, z_s, \varrho$  and  $\{\alpha_i\}_{i \in \mathbb{N}}$  such that*

$$(1.19) \quad \overline{D}(\mathbf{c}) \geq KH(\mathbf{c}|\mathcal{Q}).$$

(ii) **(Estimate for  $a_i \sim i^\gamma$  with  $\gamma < 1$ .)** *Assume that  $0 \leq \gamma < 1$  and that  $c_1$  satisfies (1.18) for some  $\delta > 0$ . If, in addition, there exists  $\beta > 1$  with*

$$(1.20) \quad M_\beta(\mathbf{c}) = \sum_{i=1}^{\infty} i^\beta c_i < +\infty$$

*then there exists  $K > 0$  depending only on  $\delta, z_s, \varrho, M_\beta(\mathbf{c})$  and  $\{\alpha_i\}_{i \in \mathbb{N}}$  such that*

$$(1.21) \quad \overline{D}(\mathbf{c}) \geq KH(\mathbf{c}|\mathcal{Q})^{\frac{\beta-\gamma}{\beta-1}}.$$

(iii) **(Estimate for small  $c_1$ .)** *Assume that  $\gamma = 1$ , or that  $0 \leq \gamma < 1$  and (1.20) holds for some  $\beta > 1$ . Assume also that for some  $\delta > 0$*

$$c_1 \leq \delta$$

*or that*

$$c_1 \geq z_s - \delta$$

*(i.e.,  $c_1$  is outside of the range given in (1.18)). Then if  $\delta > 0$  is small enough (depending only on  $\varrho$  and  $\{Q_i\}_{i \in \mathbb{N}}$ ), there exists  $\varepsilon > 0$  depending only on  $\delta, z_s, \varrho$  and  $\{\alpha_i\}_{i \in \mathbb{N}}$  if  $\gamma = 1$  (and additionally on  $M_\beta(\mathbf{c})$  if  $\gamma < 1$ ) such that*

$$(1.22) \quad \overline{D}(\mathbf{c}) \geq \varepsilon.$$

*The constants  $K$  and  $\varepsilon$  can be estimated explicitly in all cases.*



We point out that since  $\bar{z}$ ,  $z_s$ ,  $\varrho_s$ ,  $\{Q_i\}_{i \in \mathbb{N}}$  and  $\{\alpha_i\}_{i \in \mathbb{N}}$  are determined entirely by the coagulation and fragmentation coefficients and  $\varrho$ , all constants in the above theorem depend only on  $\varrho$ , the coefficients  $\{a_i\}_{i \in \mathbb{N}}$ ,  $\{b_i\}_{i \in \mathbb{N}}$ , and the additional bounds  $\delta$  or  $M_\beta$ .

The case (i) of Theorem 1.1 is optimal in the following sense:

**Theorem 1.2.** *Call  $X_\varrho$  the set of nonnegative sequences  $\mathbf{c} = \{c_i\}_{i \in \mathbb{N}}$  with mass  $\varrho$  (i.e., such that  $\sum_{i=1}^\infty i c_i = \varrho$ ). Then, there exist  $\{a_i\}_{i \in \mathbb{N}}$ ,  $\{Q_i\}_{i \in \mathbb{N}}$  that satisfy Hypotheses 1-3 with  $\gamma < 1$  such that*

$$\inf_{X_\varrho} \frac{D(\mathbf{c})}{H(\mathbf{c}|\mathcal{Q})} = 0.$$

for any  $\varrho < \varrho_s$ .

Of course, this shows that a linear inequality as that of Theorem 1.1 (i) cannot hold if  $a_i \sim i^\gamma$  with  $\gamma < 1$ .

The idea behind the proof of Theorem 1.1 is to use a discrete logarithmic Sobolev inequality with weights, motivated by works of Bobkov and Götze [6] and Barthe and Roberto [4], to show part (i). As the conditions for the validity of the log-Sobolev inequality are *not* satisfied under the conditions of part (ii), a simple interpolation is used to show the desired result in that case. Part (iii) is proved by two estimates: The case where  $c_1$  is too large follows an idea of Jabin and Niethammer, and is essentially stated already in [17], while the case where  $c_1$  is too small seems to be a new result which we provide.

From Theorem 1.1 one can conclude in a straightforward way the following theorem, our main result on the rate of convergence to equilibrium:

**Theorem 1.3.** *Let  $\{a_i\}_{i \in \mathbb{N}}$ ,  $\{Q_i\}_{i \in \mathbb{N}}$  satisfy Hypotheses 1-3 with  $0 \leq \gamma \leq 1$ , and let  $\mathbf{c}(t) = \{c_i(t)\}_{i \in \mathbb{N}}$  be a solution to the Becker-Döring equations with mass  $\varrho \in (0, \varrho_s)$ .*

(i) **(Rate for  $a_i \sim i$ .)** *If  $\gamma = 1$  then there exists a constant  $K > 0$  depending only on  $z_s, \varrho$  and  $\{\alpha_i\}_{i \in \mathbb{N}}$ , and a constant  $C > 0$  depending only on  $H(\mathbf{c}(0)|\mathcal{Q})$ ,  $z_s, \varrho$  and  $\{\alpha_i\}_{i \in \mathbb{N}}$  such that*

$$H(\mathbf{c}(t)|\mathcal{Q}) \leq C e^{-Kt} \quad \text{for } t \geq 0.$$

(ii) **(Rate for  $a_i \sim i^\gamma$ ,  $\gamma < 1$ .)** *If  $\gamma < 1$  and  $M_\beta(\mathbf{c}(0)) < +\infty$  then there exists a constant  $K > 0$  depending only on  $z_s, \varrho, M_\beta$  and  $\{\alpha_i\}_{i \in \mathbb{N}}$ , and a constant  $C > 0$  depending only on  $H(\mathbf{c}(0)|\mathcal{Q})$ ,  $z_s, \varrho, M_\beta$  and  $\{\alpha_i\}_{i \in \mathbb{N}}$  such that*

$$H(\mathbf{c}(t)|\mathcal{Q}) \leq \frac{1}{\left(C + \frac{1-\alpha}{\beta-1} Kt\right)^{\frac{\beta-1}{1-\alpha}}} \quad \text{for } t \geq 0.$$

The constants  $K$  and  $C$  can be estimated explicitly.

As remarked above, the constants  $C$  and  $K$  above depend ultimately only on the coefficients  $a_i, b_i$ , the initial mass  $\varrho$ , and the moment  $M_\beta$  in the case (ii).

There are some improvements in these theorems with respect to the existing theory. One of them is that they apply to more general initial conditions, removing the need for a finite exponential moment present in [17, 10]. Another one is that they answer the question of whether one can obtain a linear inequality such as (1.19) (i.e., whether the equivalent of Cercignani's conjecture holds), making clear the link to discrete logarithmic Sobolev inequalities. Surprisingly, it does hold in the case  $a_i \sim i$ , which is physically relevant for example in modelling polymer chains [20, 16]. As a result, the statement for  $a_i \sim i$  is quite strong: it gives full exponential convergence, with explicit constants in



terms of the parameters, with no restriction on the initial data except that of subcritical mass. Point (ii) in 1.3 also relaxes the requirements on the initial data, at the price of obtaining a slower convergence than that of [10]; we do not know whether this rate is optimal for initial conditions with polynomially decaying tails (so that  $M_\beta < \infty$  for some  $\beta > 1$ , but  $M_{\beta'} = +\infty$  for some  $\beta' > \beta$ ).

One may wonder if the method presented here can be used to reach an inequality like Jabin and Neithammer's (1.16) under the additional condition of an exponential moment. The answer is provided in the following:

**Theorem 1.4.** *Let  $\{a_i\}_{i \in \mathbb{N}}, \{Q_i\}_{i \in \mathbb{N}}$  satisfy Hypothesis 1–3 with  $0 \leq \gamma < 1$ . Let  $\mathbf{c} = \{c_i\}_{i \in \mathbb{N}}$  be an arbitrary positive sequence with mass  $\varrho \in (0, \varrho_s)$  for which there exists  $\mu > 0$  such that*

$$(1.23) \quad M_\mu^{\text{exp}}(\mathbf{c}) = \sum_{i=1}^{\infty} e^{\mu i} c_i < +\infty.$$

*Then:*

- (i) **(functional inequality.)** *There exist  $K_1, K_2, \varepsilon > 0$  depending only on  $z_s, \varrho, M_\mu^{\text{exp}}(\mathbf{c})$  and  $\{\alpha_i\}_{i \in \mathbb{N}}$  such that*

$$(1.24) \quad \overline{D}(\mathbf{c}) \geq \min \left( \frac{K_1 H(\mathbf{c}|\mathcal{Q})}{|\log(K_2 H(\mathbf{c}|\mathcal{Q}))|^{1-\gamma}}, \varepsilon \right).$$

*Moreover,  $K_1, K_2$  and  $\varepsilon$  can be given explicitly.*

- (ii) **(rate of convergence.)** *If  $\mathbf{c}(t) = \{c_i(t)\}_{i \in \mathbb{N}}$  is a solution to the Becker-Döring equations with mass  $0 < \varrho < \varrho_s$  such that there exists  $\mu > 0$  with  $M_\mu^{\text{exp}}(\mathbf{c}(0)) < +\infty$ , then there exists a constant  $K > 0$  depending only on  $z_s, \varrho, M_\mu^{\text{exp}}(\mathbf{c}(0))$  and  $\{\alpha_i\}_{i \in \mathbb{N}}$ , and a constant  $C > 0$  depending only on  $H(\mathbf{c}(0)|\mathcal{Q}), z_s, \varrho, M_\mu^{\text{exp}}(\mathbf{c}(0))$  and  $\{\alpha_i\}_{i \in \mathbb{N}}$  such that*

$$H(\mathbf{c}(t)|\mathcal{Q}) \leq C e^{-K t^{\frac{1}{2-\gamma}}}.$$

*Moreover,  $K$  and  $C$  can be given explicitly.*

**1.5. Organisation of the Paper.** The structure of the paper is as follows: In Section 2 we will present our main technical tool, a discrete version of the log-Sobolev inequality with weights. Section 3 contains the proof of Theorem 1.1 and uses Section 2 to show the first part of the theorem. We also show in this section that this method is optimal and that Cercignani's conjecture cannot hold when  $\gamma < 1$ , proving Theorem 1.2 and explore the additional inequality that appears under the assumption of a finite exponential moment. Section 4 deals with the consequences of our functional inequalities for the solutions to the Becker-Döring equation and contains the proof of Theorem 1.3 and part (ii) of Theorem 1.4. In Section 5 we briefly point out some consequences of our results for general coagulation and fragmentation equations and remark on the difficulties of obtaining stronger results in this general setting. Lastly, we conclude this work with Section 6 where we discuss a few final remarks, followed by appendices where we give proofs to some technical lemmas.

## 2. A DISCRETE WEIGHTED LOGARITHMIC SOBOLEV INEQUALITY

The key ingredient in proving Cercignani's conjecture for the Becker-Döring equations in the setting of Theorem 1.1 relies heavily on a discrete log-Sobolev inequality with weights. The theory presented here follows closely the work done by Bobkov and Götze in [6], and that of Barthe and Roberto in [4], and can be seen as a simple discrete version

of the aforementioned papers. We will give a sketch of the proof here, and in Appendix A, for the sake of completion. The reader that is familiar with these works is advised to skip this subsection and see how we use the main result proved in it in the next subsection.

**Definition 2.1.** We say that  $\boldsymbol{\mu} \in P(\mathbb{N})$  if  $\boldsymbol{\mu} = \{\mu_i\}_{i \in \mathbb{N}}$  is a non-negative sequence such that

$$\sum_{i=1}^{\infty} \mu_i = 1.$$

For any non-negative sequence  $\mathbf{g} = \{g_i\}_{i \in \mathbb{N}}$  with

$$\sum_{i=1}^{\infty} \mu_i g_i < +\infty$$

we define its *entropy* with respect to  $\boldsymbol{\mu}$  as

$$(2.1) \quad \mathbf{Ent}_{\boldsymbol{\mu}}(\mathbf{g}) = \sum_{i=1}^{\infty} \mu_i g_i \log \frac{g_i}{\sum_{i=1}^{\infty} \mu_i g_i}.$$

**Definition 2.2.** Given  $\boldsymbol{\mu} \in P(\mathbb{N})$  and positive sequence  $\boldsymbol{\nu} = \{\nu_i\}_{i \in \mathbb{N}}$  (not necessarily normalised) we say that  $\boldsymbol{\nu}$  admits a log-Sobolev inequality with respect to  $\boldsymbol{\mu}$  with constant  $0 < \Lambda < +\infty$  if for any sequence  $\mathbf{f} = \{f_i\}_{i \in \mathbb{N}}$

$$(2.2) \quad \mathbf{Ent}_{\boldsymbol{\mu}}(\mathbf{f}^2) \leq \Lambda \sum_{i=1}^{\infty} \nu_i (f_{i+1} - f_i)^2,$$

where  $\mathbf{f}^2 = \{f_i^2\}_{i \in \mathbb{N}}$ .

In what follows we will always assume that  $\boldsymbol{\mu} \in P(\mathbb{N})$ . Denoting by

$$\Psi(x) = |x| \log(1 + |x|)$$

the main theorem, and its simplified corollary, that we will prove in this subsection are:

**Theorem 2.3.** *The following two conditions are equivalent:*

- (i)  $\boldsymbol{\nu}$  admits a log-Sobolev inequality with respect to  $\boldsymbol{\mu}$  with constant  $\Lambda$ .
- (ii) For any  $m \in \mathbb{N}$  such that

$$\max \left( \sum_{i=1}^{m-1} \mu_i, \sum_{i=m+1}^{\infty} \mu_i \right) < \frac{2}{3}$$

we have that

$$(2.3) \quad B_1 = \sup_{k \geq m} \frac{\sum_{i=1}^k \frac{1}{\nu_i}}{\Psi^{-1} \left( \frac{1}{\sum_{i=k+1}^{\infty} \mu_i} \right)} < +\infty.$$

Moreover, if (ii) is valid then one can choose

$$(2.4) \quad \Lambda = 40(B_2 + 4B_1),$$

$$\text{where } B_2 = \frac{\sum_{i=1}^{m-1} \frac{1}{\nu_i}}{\Psi^{-1} \left( \frac{1}{\sum_{i=1}^{m-1} \mu_i} \right)}.$$

**Corollary 2.4.** *The following two conditions are equivalent:*

- (i)  $\boldsymbol{\nu}$  admits a log-Sobolev inequality with respect to  $\boldsymbol{\mu}$  with constant  $\Lambda$ .

(ii) For any  $m \in \mathbb{N}$  such that

$$\max \left( \sum_{i=1}^{m-1} \mu_i, \sum_{i=m+1}^{\infty} \mu_i \right) < \frac{2}{3}$$

we have that

$$(2.5) \quad D_1 = \sup_{k \geq m} \left( - \sum_{i=k+1}^{\infty} \mu_i \log \left( \sum_{i=k+1}^{\infty} \mu_i \right) \right) \left( \sum_{i=1}^k \frac{1}{\nu_i} \right) < \infty.$$

Moreover, if (ii) is valid then one can choose

$$(2.6) \quad \Lambda = 120(D_2 + 4D_1),$$

where  $D_2 = \left( - \sum_{i=1}^{m-1} \mu_i \log \left( \sum_{i=1}^{m-1} \mu_i \right) \right) \left( \sum_{i=1}^{m-1} \frac{1}{\nu_i} \right)$ .

*Remark 2.5.* One can clearly see that if

$$\sup_{k \geq 1} \left( - \sum_{i=k+1}^{\infty} \mu_i \log \left( \sum_{i=k+1}^{\infty} \mu_i \right) \right) \left( \sum_{i=1}^k \frac{1}{\nu_i} \right) < \infty$$

then one has a log-Sobolev inequality of  $\nu$  with respect to  $\mu$ . However, the introduction of the 'approximate median'  $m$  allows us to have an explicit estimation on the log-Sobolev constant  $\Lambda$ .

The rest of the subsection is dedicated to the proof of the above theorem and corollary.

**Definition 2.6.** Let  $\mu \in P(\mathbb{N})$ . Given a sequence  $\mathbf{f} = \{f_i\}_{i \in \mathbb{N}}$  we define

$$(2.7) \quad \mathcal{L}(\mathbf{f}) = \sup_{\alpha \in \mathbb{R}} \mathbf{Ent}_{\mu}((\mathbf{f} + \alpha)^2)$$

where  $\mathbf{f} + \alpha = \{f_i + \alpha\}_{i \in \mathbb{N}}$ .

**Lemma 2.7.** For any sequence  $\mathbf{f}$ , we have

$$(2.8) \quad \mathbf{Ent}_{\mu}(\mathbf{f}^2) \leq \mathcal{L}(\mathbf{f}) \leq \mathbf{Ent}_{\mu}(\mathbf{f}^2) + 2 \sum_{i=1}^{\infty} \mu_i f_i^2.$$

*Remark 2.8.* This Lemma is an adaptation of the appropriate Lemma in [26]. We leave the proof of it to Appendix A.

The next step in our path is to recast the log-Sobolev inequality as a Poincaré inequality in the Orlicz space associated to  $\Psi$ .

**Definition 2.9.** Given  $\mu \in P(\mathbb{N})$  and a Young Function,  $\Sigma : [0, +\infty) \rightarrow [0, +\infty)$ , i.e. a convex function such that

$$\frac{\Sigma(x)}{x} \xrightarrow{x \rightarrow +\infty} +\infty, \quad \frac{\Sigma(x)}{x} \xrightarrow{x \rightarrow 0} 0,$$

we define the Orlicz space  $L_{\Sigma}^{(\mu)}$  as the space of all sequences  $\mathbf{f}$  such that there exists  $k > 0$  with

$$\sum_{i=1}^{\infty} \mu_i \Sigma \left( \frac{|f_i|}{k} \right) < \infty.$$

In that case we define

$$\|\mathbf{f}\|_{L_{\Sigma}^{(\mu)}} = \inf_{k > 0} \left\{ \sum_{i=1}^{\infty} \mu_i \Sigma \left( \frac{|f_i|}{k} \right) \leq 1 \right\}.$$

In what follows we will drop the superscript  $\mu$  from the Orlicz space of  $\Psi$  and its norm. Additionally we denote by  $\Phi(x) = \Psi(x^2)$  and notice that:

(2.9)

$$\|\mathbf{f}^2\|_{L_\Psi} = \inf_{k>0} \left\{ \sum_{i=1}^{\infty} \mu_i \Psi\left(\frac{f_i^2}{k}\right) \leq 1 \right\} = \left( \inf_{\sqrt{k}>0} \left\{ \sum_{i=1}^{\infty} \mu_i \Phi\left(\frac{|f|}{\sqrt{k}}\right) \leq 1 \right\} \right)^2 = \|\mathbf{f}\|_{L_\Phi}^2.$$

**Theorem 2.10.** *The following conditions are equivalent:*

- (i)  $\nu$  admits a log-Sobolev inequality with respect to  $\mu$  with constant  $\Lambda$ .
- (ii) For any sequence  $\mathbf{f}$

$$(2.10) \quad \mathcal{L}(\mathbf{f}) \leq \Lambda \sum_{i=1}^{\infty} \nu_i (f_{i+1} - f_i)^2.$$

- (iii) For any sequence  $\mathbf{f}$

$$(2.11) \quad \|\mathbf{f} - \langle \mathbf{f} \rangle\|_{L_\Phi}^2 \leq \lambda \sum_{i=1}^{\infty} \nu_i (f_{i+1} - f_i)^2.$$

where  $\langle \mathbf{f} \rangle = \sum_{i=1}^{\infty} \mu_i f_i$ .

Moreover, if (i) or (ii) are valid one can choose  $\lambda = \frac{3}{2}\Lambda$ . If (iii) is valid one can choose  $\Lambda = 5\lambda$ .

The proof of the theorem relies on the following proposition:

**Proposition 2.11.** *For any sequence  $\mathbf{f}$  one has that*

$$(2.12) \quad \frac{2}{3} \|\mathbf{f} - \langle \mathbf{f} \rangle\|_{L_\Phi}^2 \leq \mathcal{L}(\mathbf{f}) \leq 5 \|\mathbf{f} - \langle \mathbf{f} \rangle\|_{L_\Phi}^2$$

*Proof.* We start by noticing that we may assume that  $\langle \mathbf{f} \rangle = 0$  as well as  $\|\mathbf{f} - \langle \mathbf{f} \rangle\|_{L_\Phi} = 1$ . This is true as  $\mathcal{L}$  is invariant under translations and

$$\mathbf{Ent}_\mu(\alpha \mathbf{f}) = \alpha \mathbf{Ent}_\mu(\mathbf{f}).$$

Using Lemma 2.7, we find that

$$\begin{aligned} \mathcal{L}(\mathbf{f}) &\leq \mathbf{Ent}_\mu(\mathbf{f}^2) + 2 \sum_{i=1}^{\infty} \mu_i f_i^2 = \sum_{i=1}^{\infty} \mu_i f_i^2 \log(f_i^2) + 2 \sum_{i=1}^{\infty} \mu_i f_i^2 \\ &\quad - \left( \sum_{i=1}^{\infty} \mu_i f_i^2 \right) \log \left( \sum_{i=1}^{\infty} \mu_i f_i^2 \right) \\ &\leq \sum_{i=1}^{\infty} \mu_i \Phi(f_i) + h \left( \sum_{i=1}^{\infty} \mu_i f_i^2 \right), \end{aligned}$$

where  $h(x) = 2x - x \log x$  for  $x > 0$ . As  $h$  is an increasing function on  $[0, e]$  and

$$\|\mathbf{f}\|_{L_\mu^1} \leq \|\mathbf{f}\|_{L_\mu^2} \leq \sqrt{\frac{3}{2}} \|\mathbf{f}\|_{L_\Phi},$$

(see Lemma A.2 in Appendix A) we have that

$$\|\mathbf{f}\|_{L_\mu^2}^2 \leq 2.$$

Thus, as

$$\sum_{i=1}^{\infty} \mu_i \Phi(f_i) = \sum_{i=1}^{\infty} \mu_i \Phi\left(\frac{f_i}{\|\mathbf{f}\|_{L_\Phi}}\right) \leq 1,$$

we find that

$$\mathcal{L}(\mathbf{f}) \leq 1 + h(2) \leq 5,$$

proving the right hand side inequality of (2.12). To show the left hand side inequality we assume that  $\mathcal{L}(\mathbf{f}) = 2$ . By the definition of  $\mathcal{L}$  and the fact that

$$\|\mathbf{f} - \langle \mathbf{f} \rangle\|_{L_\mu^2}^2 = \frac{1}{2} \lim_{|a| \rightarrow \infty} \mathbf{Ent}_\mu((\mathbf{f} + a)^2)$$

(see Lemma A.3 in Appendix A) we know that

$$\|\mathbf{f}\|_{L_\mu^2}^2 \leq \frac{1}{2} \mathcal{L}(\mathbf{f}) = 1.$$

This implies that

$$\begin{aligned} \sum_{i=1}^{\infty} \mu_i \Phi(f_i) &\leq 1 + \sum_{i=1}^{\infty} \mu_i f_i^2 \log f_i^2 = 1 + \mathbf{Ent}_\mu(\mathbf{f}^2) + \|\mathbf{f}\|_{L_\mu^2}^2 \log \left( \|\mathbf{f}\|_{L_\mu^2}^2 \right) \\ &\leq 1 + \mathcal{L}(\mathbf{f}) = 3, \end{aligned}$$

where we have used the fact that  $x \log(1+x) \leq 1 + x \log x$  when  $x > 0$ .

Since for any  $a \geq 1$ ,  $\Phi\left(\frac{x}{\sqrt{a}}\right) = \frac{x^2}{a^2} \log\left(1 + \frac{x^2}{a^2}\right) \leq \frac{1}{a^2} \Phi(x)$ , the above implies that

$$\sum_{i=1}^{\infty} \mu_i \Phi\left(\frac{f_i}{\sqrt{3}}\right) \leq 1$$

and as such, by the definition of  $\|\cdot\|_{L_\Phi}$ , we conclude that

$$\|\mathbf{f}\|_{L_\Phi}^2 \leq 3 = \frac{3}{2} \mathcal{L}(\mathbf{f}),$$

and the proof is complete.  $\square$

*Proof of Theorem 2.10.* The equivalence of (ii) and (iii) is immediate following Proposition 2.11, which also proves the desired connection between  $\Lambda$  and  $\lambda$ . To show that (i) implies (ii) we notice that as the right hand side of (2.2) is invariant under translation. Taking the supremum over all possible translations results in (ii). The fact that (ii) implies (i) is immediate as

$$\mathbf{Ent}_\mu(\mathbf{f}^2) \leq \mathcal{L}(\mathbf{f}).$$

$\square$

This observation that the log-Sobolev inequality with weights is actually a form of a Poincaré inequality brings to mind another inequality with weights that is closely connected to the Poincaré inequality - Hardy inequality. In its discrete form, we have that

**Theorem 2.12.** *Let  $\mu$  and  $\nu$  two sequences of positive numbers and let  $m \in \mathbb{N}$ . Then, the following two conditions are equivalent:*

(i) *There exists a finite constant  $A_{1,m} \geq 0$  such that*

$$\sum_{i=m}^{\infty} \mu_i \left( \sum_{j=m}^i f_j \right)^2 \leq A_{1,m} \sum_{i=m}^{\infty} \nu_i f_i^2,$$

*for any sequence  $\mathbf{f}$ .*

(ii) *The following holds:*

$$B_{1,m} = \sup_{k \geq m} \left( \sum_{i=k}^{\infty} \mu_i \right) \left( \sum_{i=m}^k \frac{1}{\nu_i} \right) < \infty.$$

Moreover, if any of the conditions holds than  $B_{1,m} \leq A_{1,m} \leq 4B_{1,m}$ .

The proof for the case  $m = 1$  can be found in [10], and the general case follows by the same method of proof.

**Corollary 2.13.** *Let*

$$B_m^{(1)} = \sup_{k \geq m} \left( \sum_{i=k+1}^{\infty} \mu_i \right) \left( \sum_{i=m}^k \frac{1}{\nu_i} \right).$$

Then for any sequence  $\mathbf{f}$  such that  $f_m = 0$  we have that

$$(2.13) \quad \sum_{i=m}^{\infty} \mu_i f_i^2 \leq A_m^{(1)} \sum_{i=m}^{\infty} \nu_i (f_{i+1} - f_i)^2,$$

if and only if  $B_m^{(1)} < \infty$ . In that case  $B_m^{(1)} \leq A_m^{(1)} \leq 4B_m^{(1)}$ . Additionally,

$$B_{1,m} \leq B_m^{(1)} \leq B_{1,m+1}.$$

*Proof.* This follows immediately from Theorem 2.12 applied to the sequence  $g_i = f_{i+1} - f_i$  and a simple translation argument.  $\square$

Besides the above, we will also need to have a Hardy-type inequality for sums up to a fixed integer  $m$ .

**Theorem 2.14.** *Let  $\mu$  and  $\nu$  two sequences of positive numbers and let  $m \in \mathbb{N}$ . Then, for any sequence  $\mathbf{f}$  such that  $f_m = 0$  we have that if there exists  $A > 0$  such that*

$$(2.14) \quad \sum_{i=1}^{m-1} \mu_i f_i^2 \leq A \sum_{i=1}^{m-1} \nu_i (f_{i+1} - f_i)^2,$$

then  $b_{2,m} \leq A$  where

$$b_{2,m} = \sup_{k \leq m-1} \sum_{i=1}^k \mu_i \left( \sum_{j=k}^{m-1} \frac{1}{\nu_j} \right).$$

Moreover, one can always choose

$$A = B_{2,m} = \sum_{i=1}^{m-1} \mu_i \left( \sum_{j=i}^{m-1} \frac{1}{\nu_j} \right).$$

*Proof.* We start by noticing that for any  $1 \leq i \leq m-1$  we have that

$$\begin{aligned} f_i^2 &= \left[ \sum_{j=i}^{m-1} (f_{j+1} - f_j) \right]^2 \leq \left( \sum_{j=i}^{m-1} \frac{1}{\nu_j} \right) \left( \sum_{j=i}^{m-1} \nu_j (f_{j+1} - f_j)^2 \right) \\ &\leq \left( \sum_{j=i}^{m-1} \frac{1}{\nu_j} \right) \left( \sum_{j=1}^{m-1} \nu_j (f_{j+1} - f_j)^2 \right). \end{aligned}$$

Thus

$$\sum_{i=1}^{m-1} \mu_i f_i^2 \leq \left[ \sum_{i=1}^{m-1} \mu_i \left( \sum_{j=i}^{m-1} \frac{1}{\nu_j} \right) \right] \left( \sum_{j=1}^{m-1} \nu_j (f_{j+1} - f_j)^2 \right) = B_{2,m} \sum_{j=1}^{m-1} \nu_j (f_{j+1} - f_j)^2,$$

completing the second statement. Next, for any  $j \leq m-1$  we denote by

$$\sigma_j = \sum_{i=j}^{m-1} \frac{1}{\nu_i}.$$

Fix  $k \leq m-1$  and define  $\mathbf{f}^{(k)}$  to be such that  $f_i^{(k)} = \sigma_k$  when  $i \leq k$  and  $f_i^{(k)} = \sigma_i$  when  $i > k$ . We have that

$$\sum_{i=1}^{m-1} \nu_i \left( f_{i+1}^{(k)} - f_i^{(k)} \right)^2 = \sum_{i=k}^{m-1} \nu_i \left( f_{i+1}^{(k)} - f_i^{(k)} \right)^2 = \sum_{i=k}^{m-1} \frac{1}{\nu_i} = \sigma_k.$$

On the other hand

$$\sum_{i=1}^{m-1} \mu_i \left( f_i^{(k)} \right)^2 \geq \sum_{i=1}^k \mu_i \left( f_i^{(k)} \right)^2 = \sigma_k^2 \left( \sum_{i=1}^k \mu_i \right).$$

As (2.14) is valid we see that  $A \geq \left( \sum_{i=k}^{m-1} \frac{1}{\nu_i} \right) \left( \sum_{i=1}^k \mu_i \right)$  for all  $k$ . This completes the proof.  $\square$

As we can see, the expression for the constants  $B_m^{(1)}$  and  $B_m^{(2)}$  are starting to look similar to the expression appearing in (ii) of Theorem 2.3. However, we still need a few more technicalities to complete the proof.

**Theorem 2.15.** *The following conditions are equivalent:*

- (i)  $\nu$  admits a log-Sobolev inequality with respect to  $\mu$  with constant  $\Lambda$ .
- (ii) There exists  $\eta > 0$  such that, for any sequence  $\mathbf{f} = \{f_i\}$  such that  $f_m = 0$  with  $m \in \mathbb{N}$  satisfying

$$\max \left( \sum_{i=1}^{m-1} \mu_i, \sum_{i=m+1}^{\infty} \mu_i \right) < \frac{2}{3}$$

we have that

$$\left\| (\mathbf{f}^{(0)})^2 \right\|_{L_\Psi} + \left\| (\mathbf{f}^{(1)})^2 \right\|_{L_\Psi} \leq \eta \sum_{i=1}^{\infty} \nu_i (f_{i+1} - f_i)^2,$$

where  $\mathbf{f}^{(0)} = \mathbf{f} \mathbb{1}_{i < m}$  and  $\mathbf{f}^{(1)} = \mathbf{f} \mathbb{1}_{i > m}$ .

Moreover, if condition (ii) is valid one can choose  $\Lambda = 40\eta$ .

*Proof.* Using Theorem 2.10 we notice that it is enough for us to show the equivalence of conditions (ii) of our theorem and that of Theorem 2.10.

Assume, to begin with, that (ii) of Theorem 2.10 is valid. As was shown in the aforementioned theorem, this implies that

$$(2.15) \quad \|\mathbf{f} - \langle \mathbf{f} \rangle\|_{L_\Phi}^2 \leq \frac{3\Lambda}{2} \sum_{i=1}^{\infty} \nu_i (f_{i+1} - f_i)^2.$$

Due to the conditions on  $\mathbf{f}$  and the definition of  $\mathbf{f}^{(0)}$  and  $\mathbf{f}^{(1)}$  one has that

$$\begin{aligned} \|\langle \mathbf{f}^{(0)} \rangle\|_{L_\Phi} &\leq |\langle \mathbf{f}^{(0)} \rangle| \leq \|\mathbf{f}^{(0)}\|_{L_\mu^2} \sqrt{\sum_{i=1}^{m-1} \mu_i} \\ \|\langle \mathbf{f}^{(1)} \rangle\|_{L_\Phi} &\leq |\langle \mathbf{f}^{(1)} \rangle| \leq \|\mathbf{f}^{(1)}\|_{L_\mu^2} \sqrt{\sum_{i=m+1}^{\infty} \mu_i} \end{aligned}$$

(see Lemma A.4 in Appendix A). Thus

$$\|\mathbf{f}^{(0)}\|_{L_\Phi} \leq \|\mathbf{f}^{(0)} - \langle \mathbf{f}^{(0)} \rangle\|_{L_\Phi} + \|\langle \mathbf{f}^{(0)} \rangle\|_{L_\Phi} \leq \|\mathbf{f}^{(0)} - \langle \mathbf{f}^{(0)} \rangle\|_{L_\Phi} + \sqrt{\frac{3}{2} \sum_{i=1}^{m-1} \mu_i} \|\mathbf{f}^{(0)}\|_{L_\Phi},$$



implying that

$$\|\mathbf{f}^{(0)}\|_{L_\Phi} \leq \frac{1}{1 - \sqrt{\frac{3}{2} \sum_{i=1}^{m-1} \mu_i}} \|\mathbf{f}^{(0)} - \langle \mathbf{f}^{(0)} \rangle\|_{L_\Phi},$$

and similarly

$$\|\mathbf{f}^{(1)}\|_{L_\Phi} \leq \frac{1}{1 - \sqrt{\frac{3}{2} \sum_{i=m+1}^{\infty} \mu_i}} \|\mathbf{f}^{(1)} - \langle \mathbf{f}^{(1)} \rangle\|_{L_\Phi}.$$

We can conclude, by applying (2.15) to  $\mathbf{f}^{(0)}$  and  $\mathbf{f}^{(1)}$ , that

$$\begin{aligned} \|\mathbf{f}^{(0)}\|_{L_\Phi}^2 &\leq \frac{3\Lambda}{2 \left(1 - \sqrt{\frac{3}{2} \sum_{i=1}^{m-1} \mu_i}\right)^2} \sum_{i=1}^{m-1} \nu_i (f_{i+1} - f_i)^2 \\ \text{and} \quad \|\mathbf{f}^{(1)}\|_{L_\Phi}^2 &\leq \frac{3\Lambda}{2 \left(1 - \sqrt{\frac{3}{2} \sum_{i=m+1}^{\infty} \mu_i}\right)^2} \sum_{i=m}^{\infty} \nu_i (f_{i+1} - f_i)^2. \end{aligned}$$

The result now follows from (2.9).

To show the converse, we use the translation invariance of (ii) from Theorem 2.10 to assume that  $f_m = 0$ . As such we have that  $\mathbf{f} = \mathbf{f}^{(0)} + \mathbf{f}^{(1)}$ . Moreover,

$$\begin{aligned} \|\mathbf{f} - \langle \mathbf{f} \rangle\|_{L_\Phi}^2 &\leq \left( \|\mathbf{f}^{(0)} - \langle \mathbf{f}^{(0)} \rangle\|_{L_\Phi} + \|\mathbf{f}^{(1)} - \langle \mathbf{f}^{(1)} \rangle\|_{L_\Phi} \right)^2 \\ &\leq \left( \left(1 + \sqrt{\frac{3}{2}} \sqrt{\sum_{i=1}^{m-1} \mu_i}\right) \|\mathbf{f}^{(0)}\|_{L_\Phi} + \left(1 + \sqrt{\frac{3}{2}} \sqrt{\sum_{i=m+1}^{\infty} \mu_i}\right) \|\mathbf{f}^{(1)}\|_{L_\Phi} \right)^2 \\ &\leq 2 \left(1 + \sqrt{\frac{3}{2}} \sqrt{\sum_{i=1}^{m-1} \mu_i}\right)^2 \|\mathbf{f}^{(0)}\|_{L_\Phi}^2 + 2 \left(1 + \sqrt{\frac{3}{2}} \sqrt{\sum_{i=m+1}^{\infty} \mu_i}\right)^2 \|\mathbf{f}^{(1)}\|_{L_\Phi}^2 \\ &\leq 2\eta \max \left( \left(1 + \sqrt{\frac{3}{2}} \sqrt{\sum_{i=1}^{m-1} \mu_i}\right)^2, \left(1 + \sqrt{\frac{3}{2}} \sqrt{\sum_{i=m+1}^{\infty} \mu_i}\right)^2 \right) \sum_{i=1}^{\infty} \nu_i (f_{i+1} - f_i)^2 \end{aligned}$$

where we again used (2.9). This shows the desired result due to Theorem 2.10.  $\square$

We have finally gained all the tools we need to prove Theorem 2.3.

*Proof of Theorem 2.3.* Our main tool will be Theorem 2.15. It is known that

$$\|\mathbf{f}^2\|_{L_\Psi} = \sup \left\{ \sum_{i=1}^{\infty} \mu_i f_i^2 g_i ; \sum_{i=1}^{\infty} \mu_i \Xi(g_i) \leq 1 \right\},$$

where  $\Xi$  is the Young complement of  $\Psi$ . Using Corollary 2.13 we know that if  $f_m = 0$  then

$$\sum_{i=m}^{\infty} \mu_i f_i^2 g_i \leq \Lambda \sum_{i=m}^{\infty} \nu_i (f_{i+1} - f_i)^2$$

if and only if

$$B = \sup_{k \geq m} \left( \sum_{i=k+1}^{\infty} g_i \mu_i \right) \left( \sum_{i=1}^k \frac{1}{\nu_i} \right) < \infty.$$

Taking supremum over all appropriate  $\mathbf{g} = \{g_i\}$ , we find that

$$(2.16) \quad \|\mathbf{f}^2 \mathbb{1}_{i>m}\|_{L_\Psi} \leq \Lambda \sum_{i=m}^{\infty} \nu_i (f_{i+1} - f_i)^2$$

if and only if

$$B = \sup_{k \geq m} \|\mathbb{1}_{[k+1, \infty)}\|_{L_\Psi} \sum_{i=1}^k \frac{1}{\nu_i} < \infty.$$

As

$$\begin{aligned} \|\mathbb{1}_{[k+1, \infty)}\|_{L_\Psi} &= \inf_{\alpha > 0} \left\{ \sum_{i=k+1}^{\infty} \mu_i \Psi\left(\frac{1}{\alpha}\right) \leq 1 \right\} = \inf_{\alpha > 0} \left\{ \Psi\left(\frac{1}{\alpha}\right) \leq \frac{1}{\sum_{i=k+1}^{\infty} \mu_i} \right\} \\ &= \frac{1}{\Psi^{-1}\left(\frac{1}{\sum_{i=k+1}^{\infty} \mu_i}\right)} \end{aligned}$$

we find that (2.16) is equivalent to  $B_1 < \infty$ , showing that (i) implies (ii).

Conversely, using Theorem 2.14 we find that if  $f_m = 0$  then

$$\begin{aligned} \sum_{i=1}^{m-1} \mu_i f_i^2 g_i &\leq \left[ \sum_{i=1}^{m-1} \mu_i g_i \left( \sum_{j=i}^{m-1} \frac{1}{\nu_j} \right) \right] \sum_{i=1}^{m-1} \nu_i (f_{i+1} - f_i)^2 \\ &\leq \left[ \left( \sum_{i=1}^{m-1} \mu_i g_i \right) \left( \sum_{j=1}^{m-1} \frac{1}{\nu_j} \right) \right] \sum_{i=1}^{m-1} \nu_i (f_{i+1} - f_i)^2 \end{aligned}$$

and again, by taking supremum over the appropriate  $\mathbf{g}$ , we find that

$$(2.17) \quad \|\mathbf{f}^2 \mathbb{1}_{i < m}\|_{L_\Psi} \leq B_2 \sum_{i=1}^{m-1} \nu_i (f_{i+1} - f_i)^2.$$

Thus, if  $\mathbf{f} = \{f_i\}$  is a sequence such that  $f_m = 0$ , and if in addition  $B_1 < \infty$  we have that

$$\begin{aligned} \left\| (\mathbf{f}^{(0)})^2 \right\|_{L_\Psi} + \left\| (\mathbf{f}^{(1)})^2 \right\|_{L_\Psi} &\leq B_2 \sum_{i=1}^{m-1} \nu_i (f_{i+1} - f_i)^2 + 4B_1 \sum_{i=m}^{\infty} \nu_i (f_{i+1} - f_i)^2 \\ &\leq (B_2 + 4B_1) \sum_{i=1}^{\infty} \nu_i (f_{i+1} - f_i)^2, \end{aligned}$$

where we have used Corollary 2.13. We conclude, using Theorem 2.15, that if  $B_1 < \infty$  then  $\nu$  admits a log-Sobolev inequality with respect to  $\mu$  with constant  $\Lambda$  that can be chosen to be  $\Lambda = 40(B_1 + 4B_2)$ .  $\square$

We are only left with the proof of Corollary 2.4. The proof relies on the following technical lemma, whose proof is left to Appendix A:

**Lemma 2.16.** *For any  $t \geq \frac{3}{2}$  one has that*

$$\frac{1}{3} \frac{t}{\log t} \leq \Psi^{-1}(t) \leq 2 \frac{t}{\log t}.$$

*Proof of Corollary 2.4.* Due to the choice of  $m$  and Lemma 2.16 we know that  $\Psi^{-1}(t)$  and  $\frac{t}{\log t}$  are equivalent for our choice of

$$t = \frac{1}{\sum_{i=m+1}^{\infty} \mu_i}.$$

This shows the desired equivalence using Theorem 2.3. As for the last estimation, it follows immediately from the fact that

$$B_i \leq 3D_i,$$

for  $i = 1, 2$ . □

Now that we have achieved a necessary and sufficient condition to the validity of a discrete log-Sobolev inequality with weight, we will proceed to see how it can be used to prove Theorem 1.1.

### 3. ENERGY DISSIPATION INEQUALITIES

**3.1. Cercignani's Conjecture for the Becker-Döring equations.** Motivated by our previous section, the first step in trying to show the validity of Cercignani's conjecture would be to connect between the energy dissipation,  $D(\mathbf{c})$ , and a term that resembles the right hand side of (2.2). Recall that, for any non-negative sequence  $\mathbf{c} = \{c_i\}$  we defined

$$D(\mathbf{c}) = \sum_{i=1}^{\infty} a_i Q_i \left( \frac{c_1 c_i}{Q_i} - \frac{c_{i+1}}{Q_{i+1}} \right) \left( \log \left( \frac{c_1 c_i}{Q_i} \right) - \log \left( \frac{c_{i+1}}{Q_{i+1}} \right) \right)$$

and

$$\overline{D}(\mathbf{c}) = \sum_{i=1}^{\infty} a_i Q_i \left( \sqrt{\frac{c_1 c_i}{Q_i}} - \sqrt{\frac{c_{i+1}}{Q_{i+1}}} \right)^2$$

We have the following properties:

**Lemma 3.1.** *For any non-negative sequence  $\mathbf{c}$ , the following holds*

(i) *We have that*

$$(3.1) \quad \overline{D}(\mathbf{c}) \leq D(\mathbf{c})$$

(ii) *For any  $z > 0$  we have that*

$$(3.2) \quad D(\mathbf{c}) = \sum_{i=1}^{\infty} a_i (\mathcal{Q}_z)_i (\mathcal{Q}_z)_1 \left( \frac{c_1 c_i}{(\mathcal{Q}_z)_i (\mathcal{Q}_z)_1} - \frac{c_{i+1}}{(\mathcal{Q}_z)_{i+1}} \right) \left( \log \left( \frac{c_1 c_i}{(\mathcal{Q}_z)_i (\mathcal{Q}_z)_1} \right) - \log \left( \frac{c_{i+1}}{(\mathcal{Q}_z)_{i+1}} \right) \right)$$

and

$$(3.3) \quad \overline{D}(\mathbf{c}) = \sum_{i=1}^{\infty} a_i (\mathcal{Q}_z)_i (\mathcal{Q}_z)_1 \left( \sqrt{\frac{c_1 c_i}{(\mathcal{Q}_z)_i (\mathcal{Q}_z)_1}} - \sqrt{\frac{c_{i+1}}{(\mathcal{Q}_z)_{i+1}}} \right)^2$$

*Proof.* (i) is an immediate consequence of the inequality

$$(x - y) (\log x - \log y) \geq (\sqrt{x} - \sqrt{y})^2$$

and (ii) is immediate from the definition of  $\mathcal{Q}_z$  and the homogeneous nature of the expressions involved. □

We note that property (ii) of the above lemma gives an indication of how we may be able to find a connection between  $\overline{D}(\mathbf{c})$  and  $H(\mathbf{c}|\mathcal{Q})$ . However,  $\mathcal{Q}$  is not the only equilibrium state we need to consider. Similar to the work of Jabin and Niethammer ([17]), another 'equilibrium' state that will play an important role in what is to follow is

$$\tilde{\mathcal{Q}} = \mathcal{Q}_{c_1}.$$

Indeed, it is the *only* possible 'equilibrium' under which the right hand side of (3.3) attains a form that is suitable for the log-Sobolev theory we developed. From (3.3) we find that

$$(3.4) \quad \overline{D}(\mathbf{c}) = \sum_{i=1}^{\infty} a_i \tilde{Q}_i \tilde{Q}_1 \left( \sqrt{\frac{c_i}{\tilde{Q}_i}} - \sqrt{\frac{c_{i+1}}{\tilde{Q}_{i+1}}} \right)^2$$

**Proposition 3.2.** *For given coagulation and detailed balance coefficients,  $\{a_i\}_{i \in \mathbb{N}}$ ,  $\{Q_i\}_{i \in \mathbb{N}}$ , and a given positive sequence  $\mathbf{c}$  with finite mass  $\varrho$ , we define the following measures*

$$(3.5) \quad \mu_i = \frac{\tilde{Q}_i}{\sum_{i=1}^{\infty} \tilde{Q}_i}, \quad \nu_i := \frac{a_i \tilde{Q}_i}{\sum_{j=1}^{\infty} a_j \tilde{Q}_j}, \quad i \in \mathbb{N}.$$

Then, if  $\nu$  admits a log-Sobolev inequality with respect to  $\mu$  with constant  $\Lambda$  we have that

$$(3.6) \quad \overline{D}(\mathbf{c}) \geq \frac{c_1^3 \left( \sum_{i=1}^{\infty} a_i \tilde{Q}_i \right)}{\Lambda \left( \sum_{i=1}^{\infty} \tilde{Q}_i \right) \left( c_1^2 + 2 \left( \sum_{i=1}^{\infty} c_i \right) \left( \sum_{i=1}^{\infty} \tilde{Q}_i \right) \right)} H(\mathbf{c}|\mathcal{Q})$$

*Proof.* Denote by  $f_i = \sqrt{\frac{c_i}{\tilde{Q}_i}}$ . Because  $\nu$  admits a log-Sobolev inequality with respect to  $\mu$  with constant  $\Lambda$  we have that

$$(3.7) \quad \overline{D}(\mathbf{c}) = \left( \sum_{i=1}^{\infty} a_i \tilde{Q}_i \tilde{Q}_1 \right) \sum_{i=1}^{\infty} \nu_i (f_{i+1} - f_i)^2 \geq \frac{c_1 \left( \sum_{i=1}^{\infty} a_i \tilde{Q}_i \right)}{\Lambda} \mathbf{Ent}_{\mu}(\mathbf{f}^2).$$

Next, we notice that

$$(3.8) \quad \begin{aligned} \left( \sum_{i=1}^{\infty} \tilde{Q}_i \right) \mathbf{Ent}_{\mu}(\mathbf{f}^2) &= \sum_{i=1}^{\infty} c_i \log \frac{c_i}{\tilde{Q}_i} - \left( \sum_{i=1}^{\infty} c_i \right) \left( \log \sum_{i=1}^{\infty} c_i - \log \sum_{i=1}^{\infty} \tilde{Q}_i \right) \\ &= H(\mathbf{c}|\tilde{\mathcal{Q}}) + \sum_{i=1}^{\infty} c_i - \sum_{i=1}^{\infty} \tilde{Q}_i - \left( \sum_{i=1}^{\infty} c_i \right) \left( \log \sum_{i=1}^{\infty} c_i - \log \sum_{i=1}^{\infty} \tilde{Q}_i \right) \\ &= H(\mathbf{c}|\tilde{\mathcal{Q}}) - \left( \sum_{i=1}^{\infty} \tilde{Q}_i \right) \Delta \left( \frac{\sum_{i=1}^{\infty} c_i}{\sum_{i=1}^{\infty} \tilde{Q}_i} \right), \end{aligned}$$

where  $\Delta(x) = x \log x - x + 1$ . Using the fact that

$$(3.9) \quad H(\mathbf{c}|\tilde{\mathcal{Q}}) \geq H(\mathbf{c}|\mathcal{Q})$$

(see Lemma B.1 in Appendix B) and the following Csiszár-Kullback inequality

$$(3.10) \quad \mathbf{Ent}_{\mu}(\mathbf{f}^2) \geq \frac{1}{2\langle \mathbf{f}^2 \rangle} \left( \sum_{i=1}^{\infty} |f_i^2 - \langle \mathbf{f}^2 \rangle| \mu_i \right)^2,$$

where

$$\langle \mathbf{f}^2 \rangle := \sum_{i=1}^{\infty} f_i^2 \mu_i.$$

we find that in our particular setting

$$\begin{aligned}\mathbf{Ent}_\mu(\mathbf{f}^2) &\geq \frac{\sum_{i=1}^\infty \tilde{Q}_i}{2 \sum_{i=1}^\infty c_i} \left( \sum_{i=1}^\infty \left| \frac{c_i}{\sum_{i=1}^\infty \tilde{Q}_i} - \frac{\tilde{Q}_i (\sum_{i=1}^\infty c_i)}{(\sum_{i=1}^\infty \tilde{Q}_i)^2} \right| \right)^2 \\ &= \frac{\sum_{i=1}^\infty c_i}{2 \sum_{i=1}^\infty \tilde{Q}_i} \left( \sum_{i=1}^\infty \left| \frac{c_i}{\sum_{i=1}^\infty c_i} - \frac{\tilde{Q}_i}{\sum_{i=1}^\infty \tilde{Q}_i} \right| \right)^2\end{aligned}$$

and keeping only the first term in the last sum we get

$$\mathbf{Ent}_\mu(\mathbf{f}^2) \geq \frac{\sum_{i=1}^\infty c_i}{2 \sum_{i=1}^\infty \tilde{Q}_i} \left( \left| \frac{c_1}{\sum_{i=1}^\infty c_i} - \frac{\tilde{Q}_1}{\sum_{i=1}^\infty \tilde{Q}_i} \right| \right)^2 = \frac{c_1^2}{2 \sum_{i=1}^\infty c_i \sum_{i=1}^\infty \tilde{Q}_i} \left( 1 - \frac{\sum_{i=1}^\infty c_i}{\sum_{i=1}^\infty \tilde{Q}_i} \right)^2$$

Continuing from (3.8) and using (3.9), the above inequality and the fact that

$$\Delta(x) \leq (x-1)^2$$

we find that

$$\begin{aligned}\left( \sum_{i=1}^\infty \tilde{Q}_i \right) \mathbf{Ent}_\mu(\mathbf{f}^2) &\geq H(\mathbf{c}|\mathcal{Q}) - \left( \sum_{i=1}^\infty \tilde{Q}_i \right) \left( \frac{\sum_{i=1}^\infty c_i}{\sum_{i=1}^\infty \tilde{Q}_i} - 1 \right)^2 \\ &\geq H(\mathbf{c}|\mathcal{Q}) - \frac{2}{c_1^2} \left( \sum_{i=1}^\infty \tilde{Q}_i \right)^2 \left( \sum_{i=1}^\infty c_i \right) \mathbf{Ent}_\mu(\mathbf{f}^2).\end{aligned}$$

Thus,

$$H(\mathbf{c}|\mathcal{Q}) \leq \left( \sum_{i=1}^\infty \tilde{Q}_i \right) \left( 1 + \frac{2}{c_1^2} \left( \sum_{i=1}^\infty \tilde{Q}_i \right) \left( \sum_{i=1}^\infty c_i \right) \right) \mathbf{Ent}_\mu(\mathbf{f}^2).$$

Combining the above with (3.7) completes the proof.  $\square$

**Corollary 3.3.** *Under the conditions of Proposition 3.2 and the additional assumption that  $\varrho < \varrho_s < +\infty$  and  $c_1 < z_s$  we have that*

$$(3.11) \quad \overline{D}(\mathbf{c}) \geq \frac{a_1 z_s^2 c_1^2}{\Lambda(z_s + \varrho_s)(z_s^2 + 2\varrho(z_s + \varrho_s))} H(\mathbf{c}|\mathcal{Q})$$

*Proof.* This follows immediately from (3.6) and the following estimates

$$\begin{aligned}\sum_{i=1}^\infty \tilde{Q}_i &= \sum_{i=1}^\infty Q_i c_1^i \leq c_1 \left( 1 + \frac{1}{z_s} \sum_{i=2}^\infty Q_i z_s^i \right) < c_1 \left( 1 + \frac{\varrho_s}{z_s} \right) \\ \sum_{i=1}^\infty c_i &\leq \sum_{i=1}^\infty i c_i = \varrho\end{aligned}$$

together with  $\sum_{i=1}^\infty a_i \tilde{Q}_i \geq a_1 c_1$ .  $\square$

Corollary 3.3 shows us that as long as  $c_1$  is bounded from below, Cercignani's conjecture will follow immediately from a log-Sobolev inequality for  $\nu$  with respect to  $\mu$ , defined in Proposition 3.2. Our next Proposition will show that it is true, in a specific setting.

**Proposition 3.4.** *Let  $\{a_i\}_{i \in \mathbb{N}}, \{Q_i\}_{i \in \mathbb{N}}$  satisfy Hypothesis 1-3 with  $\gamma = 1$  and let  $\mathbf{c} = \{c_i\}_{i \in \mathbb{N}}$  be an arbitrary positive sequence with finite total density  $\varrho < \varrho_s < +\infty$ . Assume that there exists  $\delta > 0$  such that*

$$c_1 \leq \bar{z} + \delta < z_s.$$

Then, the measure  $\nu$  admits a log-Sobolev inequality with respect to the measure  $\mu$  with constant

$$(3.12) \quad \Lambda = 120 \frac{C\left(\frac{\bar{z}+\delta}{z_s}\right)}{e} \frac{z_s^3}{(z_s - \bar{z} - \delta)^3} \left( \frac{3(\bar{z} + \delta)}{z_s} + \left( 1 + \frac{2e(\bar{z} + \delta)}{z_s} \sup_k \left| \log \left( \alpha_{k+1}^{\frac{1}{k+1}} \right) \right| + \frac{e(\bar{z} + \delta)}{z_s} \log \left( \frac{z_s}{z_s - \bar{z} - \delta} \right) \right) \right)$$

where  $\mu$  and  $\nu$  were defined in Proposition 3.2 and

$$C(\eta) = 1 + \sup_{k \geq 3} \left( k \left( 1 + \log \left( \frac{k}{2} \right) \right) \eta_1^{\frac{k}{2}} \right) + \frac{2\eta}{1 - \eta},$$

for  $\eta < 1$ .

*Proof.* As mentioned in the introduction to our work, we can assume without loss of generality that  $a_i = i$ . We may also assume that  $\alpha_1$ , from Hypothesis 2, equals 1. We denote by

$$\eta = \frac{c_1}{z_s} \leq \frac{\bar{z} + \delta}{z_s} = \eta_1 < 1.$$

As

$$\tilde{Q}_i = \alpha_i z_s^{1-i} c_1^i = z_s \alpha_i \eta^i$$

we find that due to the monotonicity of  $\{\alpha_i\}_{i \in \mathbb{N}}$

$$z_s \alpha_{k+1} \eta^{k+1} = \tilde{Q}_{k+1} \leq \sum_{i=k+1}^{\infty} \tilde{Q}_i \leq z_s \eta^{k+1} \sum_{i=1}^{\infty} \alpha_{i+k} \eta^{i-1} \leq \frac{z_s \alpha_{k+1} \eta^{k+1}}{1 - \eta}.$$

As such

$$\alpha_{k+1} (1 - \eta) \eta^k \leq \sum_{i=k+1}^{\infty} \mu_i \leq \alpha_{k+1} \frac{\eta^k}{1 - \eta},$$

implying that

$$(3.13) \quad - \sum_{i=k+1}^{\infty} \mu_i \log \left( \sum_{i=k+1}^{\infty} \mu_i \right) \leq \frac{\alpha_{k+1} \eta^k}{1 - \eta} \left( k \log \left( \frac{1}{\eta} \right) - \log (\alpha_{k+1} (1 - \eta)) \right).$$

Next, we notice that as

$$\sum_{i=1}^{\infty} i y^i = \frac{y}{(1 - y)^2},$$

one has that

$$z_s \eta \leq \sum_{i=1}^{\infty} i \alpha_i z_s \eta^i = \sum_{i=1}^{\infty} a_i \tilde{Q}_i \leq z_s \frac{\eta}{(1 - \eta)^2},$$

from which we find that

$$i \alpha_i (1 - \eta)^2 \eta^{i-1} \leq \nu_i \leq i \alpha_i \eta^{i-1}.$$

We notice that for  $k \geq 3$  the monotonicity of  $\{\alpha_i\}_{i \in \mathbb{N}}$  implies that

$$k \alpha_k \eta^k \sum_{i=1}^k \frac{1}{i \alpha_i} \left( \frac{1}{\eta} \right)^i = 1 + \sum_{i=1}^{k-1} \frac{k \alpha_k}{i \alpha_i} \eta^{k-i}$$

$$\begin{aligned}
&\leq 1 + \sum_{i=1}^{k-1} \frac{k}{i} \eta^{k-i} = 1 + \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \frac{k}{i} \eta^{k-i} + \sum_{i=\lfloor \frac{k}{2} \rfloor + 1}^{k-1} \frac{k}{i} \eta^{k-i} \leq 1 + k \eta_1^{\frac{k}{2}} \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{i} + \frac{k}{\lfloor \frac{k}{2} \rfloor + 1} \sum_{j=1}^{\infty} \eta_1^j \\
&\leq 1 + k \left( 1 + \log \left( \frac{k}{2} \right) \right) \eta_1^{\frac{k}{2}} + \frac{2\eta_1}{1 - \eta_1}.
\end{aligned}$$

Using the definition of  $C(\eta)$  and the fact that  $C(\eta) > 1 + \eta$  we find that for all  $k \in \mathbb{N}$

$$k \alpha_k \eta^k \sum_{i=1}^k \frac{1}{i \alpha_i} \left( \frac{1}{\eta} \right)^i \leq C(\eta_1).$$

and as such

$$(3.14) \quad \sum_{i=1}^k \frac{1}{\nu_i} \leq C(\eta_1) \frac{\eta}{(1 - \eta)^2} \frac{1}{k \alpha_k} \left( \frac{1}{\eta} \right)^k$$

Combining the above with (3.13) yields the bound

$$\begin{aligned}
&\left( - \sum_{i=k+1}^{\infty} \mu_i \log \left( \sum_{i=k+1}^{\infty} \mu_i \right) \right) \left( \sum_{i=1}^k \frac{1}{\nu_i} \right) \\
&\leq C(\eta_1) \frac{\alpha_{k+1}}{\alpha_k} \frac{\eta}{(1 - \eta)^3} \left( \log \left( \frac{1}{\eta} \right) - \frac{1}{k} \log (\alpha_{k+1} (1 - \eta)) \right).
\end{aligned}$$

Thus, with the notation of Corollary 2.4

$$\begin{aligned}
D_1 &\leq \frac{C(\eta_1)}{(1 - \eta_1)^3} \left( \sup_{0 \leq x \leq 1} (-\eta \log(\eta)) + \eta_1 \sup_k \frac{k+1}{k} \left| \log \left( \alpha_{k+1}^{\frac{1}{k+1}} \right) \right| + \eta_1 \log \left( \frac{1}{1 - \eta_1} \right) \right) \\
&\leq \frac{C(\eta_1)}{(1 - \eta_1)^3} \left( \frac{1}{e} + 2\eta_1 \sup_k \left| \log \left( \alpha_{k+1}^{\frac{1}{k+1}} \right) \right| + \eta_1 \log \left( \frac{1}{1 - \eta_1} \right) \right),
\end{aligned}$$

As  $m$ , defined in Corollary 2.4, is always finite we conclude using the same Corollary that  $\nu$  admits a log-Sobolev inequality with respect to  $\mu$ . However, in order to estimate the constant  $\Lambda$  we still need to estimate the constant  $D_2$  in the case where  $m > 1$  (otherwise,  $D_2 = 0$ ).

Since

$$\sum_{i=m}^{\infty} \mu_i \leq \frac{\alpha_m}{1 - \eta} \eta^{m-1}$$

the requirement that  $\sum_{i=1}^{m-1} \mu_i < \frac{2}{3}$  implies that

$$\frac{1}{\alpha_{m-1} \eta^{m-1}} \leq \frac{\alpha_m}{\alpha_{m-1}} \frac{3}{(1 - \eta)} \leq \frac{3}{(1 - \eta)}.$$

Using the above along with the fact that  $m > 1$  and inequality (3.14) shows that

$$\sum_{i=1}^{m-1} \frac{1}{\nu_i} \leq 3C(\eta_1) \frac{\eta_1}{(1 - \eta_1)^3} \frac{1}{m-1} \leq 3C(\eta_1) \frac{\eta_1}{(1 - \eta_1)^3}.$$

We can conclude that

$$(3.15) \quad \left( - \sum_{i=m-1}^{\infty} \mu_i \log \left( \sum_{i=m-1}^{\infty} \mu_i \right) \right) \left( \sum_{i=1}^{m-1} \frac{1}{\nu_i} \right) \leq 3 \sup_{0 \leq x \leq 1} (-x \log x) C(\eta_1) \frac{\eta_1}{(1 - \eta_1)^3}$$



from which we conclude that

$$D_2 \leq \frac{3}{e} C(\eta_1) \frac{\eta_1}{(1 - \eta_1)^3}$$

which completes the proof, as the result follows directly from Corollary 2.4.  $\square$

We finally have all the tools to prove part (i) of Theorem 1.1:

*Proof of part (i) of Theorem 1.1.* The result follows immediately from Corollary 3.3, Proposition 3.4 and condition (1.18).  $\square$

The last part of this section will be devoted to the proof of part (ii) of Theorem 1.1. For that we will need the following lemma:

**Lemma 3.5.** *For any  $\beta \geq 0$ , any non-negative sequence  $\mathbf{c}$  and positive sequence  $\{Q_i\}_{i \geq 1}$  it holds that*

$$(3.16) \quad \sum_{i=1}^{\infty} i^{\beta} Q_i \left( \sqrt{\frac{c_1 c_i}{Q_i}} - \sqrt{\frac{c_{i+1}}{Q_{i+1}}} \right)^2 \leq 2 \left( c_1 + \sup_j \frac{Q_j}{Q_{j+1}} \right) \sum_{i=1}^{\infty} i^{\beta} c_i.$$

*Proof.* The proof is a direct result of the inequality  $(a + b)^2 \leq 2(a^2 + b^2)$ :

$$\begin{aligned} \sum_{i=1}^{\infty} i^{\beta} Q_i \left( \sqrt{\frac{c_1 c_i}{Q_i}} - \sqrt{\frac{c_{i+1}}{Q_{i+1}}} \right)^2 &\leq 2c_1 \sum_{i=1}^{\infty} i^{\beta} c_i + 2 \sum_{i=1}^{\infty} i^{\beta} \frac{Q_i}{Q_{i+1}} c_{i+1} \\ &\leq 2 \left( c_1 + \sup_j \frac{Q_j}{Q_{j+1}} \right) \sum_{i=1}^{\infty} i^{\beta} c_i. \end{aligned}$$

$\square$

*Proof of part (ii) of Theorem 1.1.* We denote by  $\overline{D}_{\gamma}(\mathbf{c})$  the lower free energy dissipation of  $\mathbf{c}$  associated to the coagulation coefficient  $a_i = i^{\gamma}$ . According to part (i) of Theorem 1.1, there exists  $K > 0$  that depends only on  $\delta, z_s, \varrho$  and  $\{\alpha_i\}_{i \in \mathbb{N}}$  such that

$$\overline{D}_1(\mathbf{c}) \geq KH(\mathbf{c}|\mathcal{Q}).$$

Using interpolation between  $\gamma$  and  $\beta$  we find that

$$(3.17) \quad \overline{D}_1(c) \leq \overline{D}_{\gamma}^{\frac{\beta-1}{\beta-\gamma}}(c) \overline{D}_{\beta}^{\frac{1-\gamma}{\beta-\gamma}}(c) \leq 2^{\frac{1-\gamma}{\beta-\gamma}} \overline{D}_{\gamma}^{\frac{\beta-1}{\beta-\gamma}}(c) \left( z_s + \frac{1}{z_s} \sup_j \frac{\alpha_j}{\alpha_{j+1}} \right)^{\frac{1-\gamma}{\beta-\gamma}} M_{\beta}^{\frac{1-\gamma}{\beta-\gamma}}$$

where we have used Lemma 3.5, the upper bound on  $c_1$  and Hypothesis (2). Therefore

$$(3.18) \quad D(\mathbf{c}) \geq \overline{D}_{\gamma}(\mathbf{c}) \geq \left( \frac{z_s K^{\frac{\beta-\gamma}{1-\gamma}}}{2 \left( z_s^2 + \sup_j \frac{\alpha_j}{\alpha_{j+1}} \right) M_{\beta}} \right)^{\frac{1-\gamma}{\beta-1}} H(\mathbf{c}|\mathcal{Q})^{\frac{\beta-\gamma}{\beta-1}}$$

and the proof is now complete.  $\square$

This concludes the part of the proof of Theorem 1.1 that relied on the log-Sobolev inequality. In the next subsection we will address the question of what happens when  $c_1$  escapes its 'good region', given by (1.18).

**3.2. Energy Dissipation Estimate when  $c_1$  is 'Far' From Equilibrium.** The goal of this subsection is to show that when  $c_1$  is far from equilibrium, in the aforementioned sense, then while we may lose our desired inequality between  $\overline{D}(\mathbf{c})$  and  $H(\mathbf{c}|\mathcal{Q})$ , the energy dissipation becomes *uniformly large* - forcing the free energy to decrease (and as a consequence, the distance between  $c_1$  and  $\bar{z}$  decreases as well).

The next proposition, dealing with the case when  $c_1$  is 'too large', is an adaptation of a theorem from [17].

**Proposition 3.6.** *Let  $\{a_i\}_{i \in \mathbb{N}}, \{Q_i\}_{i \in \mathbb{N}}$  be the coagulation and detailed balance coefficients for the Becker-Döring equations. Assume that  $\inf_i a_i > 0$  and*

$$\lim_{i \rightarrow \infty} \frac{Q_{i+1}}{Q_i} = \frac{1}{z_s}.$$

*Let  $\mathbf{c} = \{c_i\}$  be a non-negative sequence with finite total density  $\varrho < \varrho_s$ . Then, if*

$$c_1 > \bar{z} + \delta$$

*for any  $\delta > 0$ , we have that*

$$\overline{D}(\mathbf{c}) > \varepsilon_1,$$

*for a fixed constant  $\varepsilon_1$  that depends only on  $\{Q_i\}_{i \in \mathbb{N}}, \bar{z}, z_s$  and  $\delta$ .*

*Proof.* Without loss of generality we may assume that  $\bar{z} + \delta < z_s$ . Denoting by  $u_i = \frac{c_i}{Q_i}$  we notice that

$$\overline{D}(\mathbf{c}) = \sum_{i=1}^{\infty} a_i Q_i (\sqrt{c_1} u_i - \sqrt{u_{i+1}})^2.$$

Let  $\lambda < 1$  be such that  $\lambda c_1 = \bar{z} + \frac{\delta}{2}$  and let  $i_0 \in \mathbb{N}$  be the first index such that

$$u_{i+1} < \lambda c_1 u_i.$$

This index exists, else, for any  $i \in \mathbb{N}$  we have

$$(3.19) \quad u_{i+1} \geq \lambda c_1 u_i \geq (\lambda c_1)^i c_1$$

and thus

$$\varrho = \sum_{i=1}^{\infty} i c_i \geq c_1 + c_1 \sum_{i=2}^{\infty} i Q_i (\lambda c_1)^{i-1} \geq \sum_{i=1}^{\infty} i Q_i \left( \bar{z} + \frac{\delta}{2} \right)^i,$$

which is a contradiction.

Due to the positivity of each term in the sum consisting of the lower free energy dissipation, we conclude that

$$(3.20) \quad \overline{D}(\mathbf{c}) \geq a_{i_0} Q_{i_0} \left( 1 - \sqrt{\lambda} \right)^2 c_1 u_{i_0} \geq a_{i_0} Q_{i_0} \lambda^{i_0-1} c_1^{i_0+1} \left( 1 - \sqrt{\lambda} \right)^2$$

where we have used the fact that up to  $i_0 - 1$  we have inequality (3.19).

As we know that there exists  $C > 0$ , depending only on  $\{Q_i\}_{i \in \mathbb{N}}, \bar{z}, z_s$  and  $\delta$  such that

$$\sum_{i=i_0+1}^{\infty} i c_1 (\lambda c_1)^{i-1} Q_i \leq C Q_{i_0} (\lambda c_1)^{i_0} c_1$$

(see Lemma B.2 in Appendix B), we conclude that, using (3.19) again,

$$C Q_{i_0} (\lambda c_1)^{i_0} c_1 \geq \tilde{\varrho} - \sum_{i=1}^{i_0} i Q_i (\lambda c_1)^{i-1} c_1 \geq \tilde{\varrho} - \sum_{i=1}^{i_0} i c_i \geq \tilde{\varrho} - \varrho,$$

where  $\tilde{\varrho} = \sum_{i=1}^{\infty} i Q_i (\lambda c_1)^{i-1} c_1$ . We can estimate the difference  $\varrho - \tilde{\varrho}$  as

$$\tilde{\varrho} - \varrho \geq \sum_{i=1}^{\infty} i Q_i \left( \left( \bar{z} + \frac{\delta}{2} \right)^i - \bar{z}^i \right) \geq \left( \sum_{i=1}^{\infty} i^2 Q_i \bar{z}^{i-1} \right) \frac{\delta}{2}.$$

In conclusion, there exists a universal constant  $C_1 > 0$ , depending only on  $\{Q_i\}_{i \in \mathbb{N}}$ ,  $\bar{z}$ ,  $z_s$  and  $\delta$ , and not on  $i_0$ ,  $c_1$  or  $\lambda$ , such that

$$Q_{i_0} (\lambda c_1)^{i_0} c_1 > C_1.$$

Recalling (3.20) and using the fact that  $\lambda = \frac{\bar{z} + \frac{\delta}{2}}{c_1} < \frac{\bar{z} + \frac{\delta}{2}}{\bar{z} + \delta}$  we find that:

$$\overline{D}(\mathbf{c}) \geq C_1 a_{i_0} \frac{(1 - \sqrt{\lambda})^2}{\lambda} \geq C_1 \inf_{i \geq 1} a_i \frac{\left( \sqrt{\bar{z} + \delta} - \sqrt{\bar{z} + \frac{\delta}{2}} \right)^2}{\bar{z} + \frac{\delta}{2}},$$

completing the proof.  $\square$

Next, we present a new lower bound estimation for the energy dissipation in the case where  $c_1$  is 'too small'.

**Lemma 3.7.** *Let  $\{a_i\}_{i \in \mathbb{N}}$ ,  $\{Q_i\}_{i \in \mathbb{N}}$  be the coagulation and detailed balance coefficients for the Becker-Döring equations. Assume that*

$$\begin{aligned} \overline{Q} &= \sup_i \frac{Q_i}{Q_{i+1}} < +\infty & \underline{Q} &= \inf_i \frac{Q_i}{Q_{i+1}} < +\infty \\ \overline{a} &= \sup_i \frac{a_i}{a_{i+1}} < +\infty & \underline{a} &= \inf_i \frac{a_i}{a_{i+1}} < +\infty, \end{aligned}$$

and let  $\mathbf{c}$  be a non-negative sequence such that

$$c_1 < \delta$$

for some  $\delta > 0$ . Then,

$$\overline{D}(\mathbf{c}) \geq \underline{Q} \underline{a} \left( \sum_{i=1}^{\infty} a_i c_i - a_1 \delta \right) - 2\sqrt{\delta} \sqrt{\overline{Q} \overline{a}} \left( \sum_{i=1}^{\infty} a_i c_i \right).$$

*Proof.* Expanding the square, one has

$$\overline{D}(\mathbf{c}) = c_1 \sum_{i=1}^{\infty} a_i c_i + \sum_{i=1}^{\infty} a_i \frac{Q_i}{Q_{i+1}} c_{i+1} - 2\sqrt{c_1} \sum_{i=1}^{\infty} a_i \sqrt{\frac{Q_i}{Q_{i+1}}} \sqrt{c_i c_{i+1}}$$

so that

$$\begin{aligned} \overline{D}(\mathbf{c}) &\geq \underline{Q} \underline{a} \left( \sum_{i=2}^{\infty} a_i c_i \right) - 2\sqrt{c_1} \sqrt{\overline{Q} \overline{a}} \sqrt{\sum_{i=2}^{\infty} a_i c_i} \sqrt{\sum_{i=1}^{\infty} a_i c_i} \\ &\geq \underline{Q} \underline{a} \left( \sum_{i=1}^{\infty} a_i c_i - a_1 \delta \right) - 2\sqrt{\delta} \sqrt{\overline{Q} \overline{a}} \left( \sum_{i=1}^{\infty} a_i c_i \right), \end{aligned}$$

which is the desired result.  $\square$

**Proposition 3.8.** *Let  $\{a_i\}_{i \in \mathbb{N}}, \{Q_i\}_{i \in \mathbb{N}}$  be the coagulation and detailed balance coefficients for the Becker-Döring equations. Assume that*

$$\overline{Q} = \sup_i \frac{Q_i}{Q_{i+1}} < +\infty \quad \underline{Q} = \inf_i \frac{Q_i}{Q_{i+1}} < +\infty.$$

*Let  $\mathbf{c}$  be a non-negative sequence with finite total density  $\varrho$ . Then:*

- (i) *If  $a_i = i$  then there exists a  $\delta_1 > 0$ , depending only on  $\overline{Q}, \underline{Q}$  and  $\varrho$  such that if  $c_1 < \delta_1$  then*

$$\overline{D}(\mathbf{c}) \geq \frac{Q\varrho}{4}.$$

- (ii) *If  $a_i = i^\gamma$  for  $\gamma < 1$  and there exists  $\beta > 1$  such that  $M_\beta < +\infty$ , then there exists  $\delta_1 > 0$ , depending only on  $\overline{Q}, \underline{Q}, \varrho$  and  $M_\beta$  such that if  $c_1 < \delta_1$  then*

$$\overline{D}(\mathbf{c}) \geq \frac{Q\varrho^{\frac{\beta-\gamma}{\beta-1}}}{4M_\beta^{\frac{1-\gamma}{\beta-1}}}.$$

*Proof.* Both (i) and (ii) will follow immediately from Lemma 3.7 and a suitable choice of  $\delta_1$ . Indeed, for (i) we notice that

$$\underline{Q}\underline{a} \left( \sum_{i=1}^{\infty} a_i c_i - a_1 \delta \right) - 2\sqrt{\delta} \sqrt{\overline{Q}\underline{a}} \left( \sum_{i=1}^{\infty} a_i c_i \right) = \frac{Q}{2} (\varrho - \delta) - 2\sqrt{\delta} \sqrt{\overline{Q}\varrho},$$

where we have used the notations of Lemma 3.7. As the above is less than  $\frac{Q\varrho}{2}$  and converges to it as  $\delta$  goes to zero, we can find  $\delta_1$  that satisfies the desired result.

For (ii) we notice that the following interpolation estimate

$$\varrho = \sum_{i=1}^{\infty} i c_i \leq \left( \sum_{i=1}^{\infty} i^\gamma c_i \right)^{\frac{\beta-1}{\beta-\gamma}} (M_\beta)^{\frac{1-\gamma}{\beta-\gamma}}$$

along with the fact that  $\sum_{i=1}^{\infty} i^\gamma c_i \leq \varrho$  implies that

$$\underline{Q}\underline{a} \left( \sum_{i=1}^{\infty} a_i c_i - a_1 \delta \right) - 2\sqrt{\delta} \sqrt{\overline{Q}\underline{a}} \left( \sum_{i=1}^{\infty} a_i c_i \right) \geq \frac{Q}{2} \left( \frac{\varrho^{\frac{\beta-\gamma}{\beta-1}}}{M_\beta^{\frac{1-\gamma}{\beta-1}}} - \delta \right) - 2\sqrt{\delta} \sqrt{\overline{Q}\varrho},$$

from which the result follows. □

We are finally ready to complete the proof of Theorem 1.1:

*Proof of part (iii) of Theorem 1.1.* This follows immediately from Propositions 3.6 and 3.8. □

Now that we have our general functional inequality at hand one may wonder how sharp is this method of using the log-Sobolev inequality? Perhaps we were too coarse in our estimation, and Cercignani's conjecture is valid in the case  $a_i = i^\gamma$  with  $\gamma < 1$  under the restrictions of Theorem 1.1. The answer, surprisingly, is that this method is optimal, as we shall see in the next subsection.

**3.3. Optimality of the Results.** This subsection is devoted to showing that unlike the case  $a_i = i$ , the case  $a_i = i^\gamma$  when  $\gamma < 1$  admits no Cercignani's Conjecture, even if  $c_1$  is bounded appropriately. This is stated in Theorem 1.2.

*Proof of Theorem 1.2.* We start by choosing  $a_i = i^\gamma$ ,  $\gamma < 1$ , and  $Q_i = e^{-\lambda(i-1)}$  ( $i \geq 1$ ) for some  $\lambda \geq 0$ . We will show the desired result by constructing a family of non-negative sequences,  $\{\mathbf{c}^{(\varepsilon)}\}_{\varepsilon > 0}$  with a fixed mass  $\varrho$  such that

$$\lim_{\varepsilon \rightarrow 0} \frac{D(\mathbf{c}^{(\varepsilon)})}{H(\mathbf{c}^{(\varepsilon)}|\mathcal{Q})} = 0.$$

Let  $\xi > 0$  be such that

$$\frac{\varrho}{2} = \sum_{i=1}^{\infty} i e^\lambda e^{-\xi i} = \frac{e^{\lambda-\xi}}{(1-e^{-\xi})^2}.$$

Consider the sequence  $\mathbf{c}^{(\varepsilon)} = \{c_i^{(\varepsilon)}\}$  given by

$$c_i^{(\varepsilon)} = e^\lambda e^{-\xi i} + A_\varepsilon e^{-\varepsilon i}, \quad i \in \mathbb{N}$$

where  $0 < \varepsilon$  is small and  $A_\varepsilon$  is chosen such that the mass of the sequence  $\mathbf{c}^{(\varepsilon)}$  is  $\varrho$ , i.e.  $A_\varepsilon = \frac{\varrho}{2} e^\varepsilon (1 - e^{-\varepsilon})^2$ . Next, as  $\frac{Q_i}{Q_{i+1}} = e^\lambda$  for any  $i \geq 1$ , we see that

$$\begin{aligned} \frac{Q_i}{Q_{i+1}} c_{i+1}^{(\varepsilon)} - c_1^{(\varepsilon)} c_i^{(\varepsilon)} &= e^{2\lambda} e^{-\xi(i+1)} + A_\varepsilon e^\lambda e^{-\varepsilon(i+1)} - e^{2\lambda} e^{-\xi(i+1)} - A_\varepsilon e^\lambda (e^{-\xi i - \varepsilon} + e^{-\varepsilon i - \xi}) - A_\varepsilon^2 e^{-\varepsilon(i+1)} \\ &= A_\varepsilon e^\lambda e^{-\varepsilon(i+1)} (1 - e^{-(\xi-\varepsilon)} - e^{-(\xi-\varepsilon)i} - A_\varepsilon e^{-\lambda}) > 0 \end{aligned}$$

for  $\varepsilon$  small enough depending only on  $\lambda, \xi$  and  $\varrho$  but not on  $i$ . Additionally, one can easily verify that

$$\frac{Q_i c_{i+1}^{(\varepsilon)}}{Q_{i+1} c_1^{(\varepsilon)} c_i^{(\varepsilon)}} \leq e^\lambda \left(1 + \frac{1}{A_\varepsilon}\right).$$

As such, setting  $B_{z,\gamma} = \sum_{i=1}^{\infty} i^\gamma e^{-zi}$  for any  $z > 0$ , we find that

$$\begin{aligned} D(\mathbf{c}^{(\varepsilon)}) &= \sum_{i=1}^{\infty} i^\gamma \left( \frac{Q_i}{Q_{i+1}} c_{i+1}^{(\varepsilon)} - c_i^{(\varepsilon)} \right) \log \left( \frac{Q_i c_{i+1}^{(\varepsilon)}}{Q_{i+1} c_1^{(\varepsilon)} c_i^{(\varepsilon)}} \right) \\ (3.21) \quad &\leq A_\varepsilon e^\lambda B_{\varepsilon,\gamma} \log \left( e^\lambda \left(1 + \frac{1}{A_\varepsilon}\right) \right) ((1 - A_\varepsilon e^{-\lambda}) e^{-\varepsilon} - e^{-\xi}) \\ &\quad - A_\varepsilon e^\lambda B_{\xi,\gamma} \log \left( e^{\lambda-\varepsilon} \left(1 + \frac{1}{A_\varepsilon}\right) \right). \end{aligned}$$

As  $A_\varepsilon \approx \frac{\varrho}{2} \varepsilon^2$  when  $\varepsilon$  approaches zero, and  $B_{\varepsilon,\gamma}$  is of order  $\varepsilon^{-(1+\gamma)}$  (see Lemma B.3 in Appendix B) we conclude that

$$\lim_{\varepsilon \rightarrow 0} D(\mathbf{c}^{(\varepsilon)}) = 0.$$

Lastly, we turn our attention to the relative free energy. We start by denoting by  $\bar{\xi} > 0$  the unique parameter for which

$$\varrho = e^\lambda \sum_{i=1}^{\infty} i e^{-\bar{\xi} i}.$$

Clearly,  $\bar{\xi} < \xi$  and the associated equilibrium with mass  $\varrho$  is  $Q_i = e^\lambda e^{-\bar{\xi} i}$ . Since, for any fixed  $i \geq 1$ , it holds

$$\lim_{\varepsilon \rightarrow 0} c_i^{(\varepsilon)} = c_i^{(0)} = e^\lambda e^{-\xi i}$$

using Fatou's lemma we can conclude that

$$\liminf_{\varepsilon \rightarrow 0} H(c^{(\varepsilon)} | \mathcal{Q}) \geq H(c^{(0)} | \mathcal{Q}) > 0$$

as  $c^{(0)} \neq \mathcal{Q}$ . □

*Remark 3.9.* We Notice the following:

- In the example we provided  $z_s = e^\lambda < +\infty$  but  $\varrho_s = +\infty$ . This, however, is not a great obstacle as all our proofs rely on some *positive* distance from  $z_s$  and  $\varrho_s$ , and can be reformulated accordingly.
- The constructed sequence  $c^{(\varepsilon)}$  satisfies

$$\sup_{\varepsilon} \sum_{i=1}^{\infty} i^{\beta} c_i^{(\varepsilon)} = +\infty$$

for any  $\beta > 1$ . Thus, the conclusion of part (ii) of Theorem 1.1 does not apply to it. Actually, one can easily check that  $\lim_{\varepsilon \rightarrow 0} \frac{D(c^{(\varepsilon)})}{(H(c^{(\varepsilon)} | \mathcal{Q}))^s} = 0$  for any  $s > 0$ .

**3.4. Inequalities with Exponential Moments.** Up to now, we have avoided using exponential moments in any of our functional inequalities. In this section we will show that when  $0 \leq \gamma < 1$ , under the additional assumption of a bounded exponential moment, one can obtain a far better functional inequality between  $\overline{D}(c)$  and  $H(c | \mathcal{Q})$ , extending the result given by Jabin and Neithammer.

The key idea in this section is to avoid using the interpolation inequality (3.17) and replace it with one that involved an exponential weight.

**Proposition 3.10.** *Let  $\mathbf{f}$  be a non-negative sequence and let  $0 \leq \gamma < 1$ . Assume that there exists  $\mu \in (0, 4 \log 2)$  such that*

$$\sum_{i=1}^{\infty} e^{\mu i} f_i = M_{\mu}^{\exp}(\mathbf{f}) < +\infty.$$

*Then,*

$$(3.22) \quad M_{\gamma}(\mathbf{f}) \geq \frac{M_1(\mathbf{f})}{2 \left( \frac{2}{\mu} \log \left( \frac{4M_{\mu}^{\exp}(\mathbf{f})}{\mu e M_1(\mathbf{f})} \right) \right)^{1-\gamma}}$$

*where  $M_{\alpha}(\mathbf{f})$  denotes the  $\alpha$ -moment of  $\mathbf{f}$ .*

*Proof.* For simplicity, we will use the notation of  $M_1$  and  $M_{\mu}^{\exp}$  instead of  $M_1(\mathbf{f})$  and  $M_{\mu}^{\exp}(\mathbf{f})$ . We start with the simple inequality

$$(3.23) \quad \begin{aligned} M_1 &= \sum_{i=1}^{\infty} i f_i = \sum_{i=1}^N i^{1-\gamma} i^{\gamma} f_i + \sum_{i=N+1}^{\infty} i e^{-\frac{\mu i}{2}} e^{-\frac{\mu i}{2}} e^{\mu i} f_i \\ &\leq N^{1-\gamma} M_{\gamma} + \frac{2e^{-\frac{\mu(N+1)}{2}}}{\mu e} M_{\mu}^{\exp}, \quad \forall N \in \mathbb{N} \end{aligned}$$

where we used the fact that  $\sup_{x \geq 0} x e^{-\lambda x} = \frac{1}{\lambda e}$  for any  $\lambda > 0$ . Our goal will be to choose a particular  $N$  to plug in the inequality above to conclude the desired result. Again, using the supremum of  $g(x) = x e^{-\lambda x}$ , we conclude that

$$M_1 \leq \frac{1}{\mu e} M_{\mu}^{\exp}.$$

As  $\mu < 4 \log 2$  we find that

$$M_1 < \frac{4M_\mu^{\text{exp}}}{\mu e^{1+\frac{\mu}{2}}}.$$

from which we conclude that  $N = \left\lfloor \frac{2}{\mu} \log \left( \frac{4M_\mu^{\text{exp}}}{\mu e M_1} \right) \right\rfloor \geq 1$ . Plugging this  $N$  into (3.23) we see that  $e^{-\frac{\mu(N+1)}{2}} \leq \frac{\mu e M_1}{4M_\mu^{\text{exp}}}$ , and as such

$$M_\gamma \geq N^{\gamma-1} \frac{M_1}{2}$$

and the result follows.  $\square$

With this proposition at hand, we are prepared to show part (i) of Theorem 1.4.

*Proof of part (i) of Theorem 1.4.* Without loss of generality we may assume that  $\mu \in (0, 4 \log 2)$ . Introduce the sequence  $\mathbf{f} = \{f_i\}$  where

$$f_i = Q_i \left( \sqrt{\frac{c_1 c_i}{Q_i}} - \sqrt{\frac{c_{i+1}}{Q_{i+1}}} \right)^2, \quad i \geq 1.$$

Following the same proof as presented in Lemma 3.5 we find that

$$M_\mu^{\text{exp}}(\mathbf{f}) \leq 2 \left( c_1 + z_s \sup_j \frac{\alpha_j}{\alpha_{j+1}} \right) M_\mu^{\text{exp}}(\mathbf{c}).$$

Thus, using the simple fact that  $M_\alpha(\mathbf{f}) = \overline{D}_\alpha(\mathbf{c})$ , for any  $\alpha > 0$ , together with Proposition 3.10 and parts (i) and (iii) of Theorem 1.1 yield the desired functional inequality.  $\square$

#### 4. RATE OF CONVERGENCE TO EQUILIBRIUM

In this section we will use all the information we gathered so far to show the proof of Theorems 1.3 and part (ii) of Theorem 1.4, giving an explicit rate of convergence to equilibrium for the Becker-Döring equations.

The convergence is an immediate consequence of Theorem 1.1 and part (i) of Theorem 1.4, yet we provide a proof here for the sake of completion and to show that we can find all the constants explicitly.

*Proof of Theorem 1.3.* Due to Theorem 1.1 we conclude the following differential inequality:

$$(4.1) \quad \frac{d}{dt} H(\mathbf{c}(t)|\mathcal{Q}) \leq \begin{cases} -\min(KH(\mathbf{c}(t)|\mathcal{Q}), \varepsilon) & \gamma = 1, \\ -\min\left(KH(\mathbf{c}(t)|\mathcal{Q})^{\frac{\beta-\gamma}{\beta-1}}, \varepsilon\right) & 0 \leq \gamma < 1, \end{cases}$$

for appropriate  $K$  and  $\varepsilon$ . We claim that there exists  $t_0 \geq 0$  such that for all  $t \geq t_0$

$$(4.2) \quad H(\mathbf{c}(t)|\mathcal{Q}) \leq \begin{cases} \frac{\varepsilon}{K} & \gamma = 1 \\ \left(\frac{\varepsilon}{K}\right)^{\frac{\beta-1}{\beta-\gamma}} & 0 \leq \gamma < 1. \end{cases}$$

Indeed, if  $H(\mathbf{c}(t)|\mathcal{Q})$  is bigger then the appropriate constants in  $[0, t]$  then

$$\frac{d}{ds} H(\mathbf{c}(s)|\mathcal{Q}) \leq -\varepsilon \quad \forall s \in (0, t),$$

implying that

$$H(\mathbf{c}(t)|\mathcal{Q}) \leq H(\mathbf{c}(0)|\mathcal{Q}) - \varepsilon t.$$



We define

$$t_0 = \begin{cases} \min \left( 0, \frac{H(\mathbf{c}(0)|\mathcal{Q}) - \frac{\varepsilon}{K}}{\varepsilon} \right) & \gamma = 1 \\ \min \left( 0, \frac{H(\mathbf{c}(0)|\mathcal{Q}) - \left(\frac{\varepsilon}{K}\right)^{\frac{\beta-1}{\beta-\gamma}}}{\varepsilon} \right) & 0 \leq \gamma < 1. \end{cases}$$

and find that  $H(\mathbf{c}(t_0)|\mathcal{Q})$  satisfies the appropriate inequality in (4.2). As  $H(\mathbf{c}(t)|\mathcal{Q})$  is decreasing, we conclude that (4.2) is valid for any  $t \geq t_0$ .

With this in hand, along with (4.1), we have that for all  $t \geq t_0$ :

$$H(\mathbf{c}(t)|\mathcal{Q}) \leq \begin{cases} H(\mathbf{c}(t_0)|\mathcal{Q})e^{-K(t-t_0)} & \gamma = 1 \\ \frac{1}{\left( H(\mathbf{c}(t_0)|\mathcal{Q})^{\frac{\gamma-1}{\beta-1}} + \frac{1-\gamma}{\beta-1} K(t-t_0) \right)^{\frac{\beta-1}{1-\gamma}}} & 0 \leq \gamma < 1. \end{cases}$$

As

$$H(\mathbf{c}(t_0)|\mathcal{Q}) = \begin{cases} \min \left( H(\mathbf{c}(0)|\mathcal{Q}), \frac{\varepsilon}{K} \right) & \gamma = 1 \\ \min \left( H(\mathbf{c}(0)|\mathcal{Q}), \left(\frac{\varepsilon}{K}\right)^{\frac{\beta-1}{\beta-\gamma}} \right) & 0 \leq \gamma < 1, \end{cases}$$

and  $t_0$  is given explicitly we conclude that

$$C(H(\mathbf{c}(0)|\mathcal{Q})) = \begin{cases} H(\mathbf{c}(0)|\mathcal{Q}) & \gamma = 1, t_0 = 0 \\ \frac{\varepsilon}{K} e^{K \frac{H(\mathbf{c}(0)|\mathcal{Q}) - \frac{\varepsilon}{K}}{\varepsilon}} & \gamma = 1, t_0 > 0 \\ H(\mathbf{c}(0)|\mathcal{Q}) & 0 \leq \gamma < 1, t_0 = 0 \\ \left(\frac{\varepsilon}{K}\right)^{\frac{\gamma-1}{\beta-\gamma}} - \frac{1-\gamma}{\beta-1} K \frac{H(\mathbf{c}(0)|\mathcal{Q}) - \left(\frac{\varepsilon}{K}\right)^{\frac{\beta-1}{\beta-\gamma}}}{\varepsilon} & 0 \leq \gamma < 1, t_0 > 0, \end{cases}$$

completing the proof.  $\square$

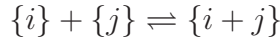
*Proof of part (ii) of Theorem 1.4.* This follows from part (i) of Theorem 1.4 by the same methods used in the above proof and the fact that

$$\sup_{t \geq 0} M_{\mu'}^{\text{exp}}(\mathbf{c}(t)) < +\infty$$

for some  $0 < \mu' < \mu$  (a known result from [17]).  $\square$

## 5. CONSEQUENCES FOR GENERAL COAGULATION AND FRAGMENTATION MODELS

The Becker-Döring equations (1.1) are derived under the assumption that the only relevant reactions taking place are those between monomers and clusters of any size. One can obtain a more general model by taking into account reactions between clusters of any size. Keeping the notation of the introduction, this means that we consider reactions of the type



for any positive integer sizes  $i$  and  $j$ . We assume their coagulation rate (i.e., the reaction from left to right) is determined by a coefficient we call  $a_{i,j}$ , and their fragmentation rate (the reaction from right to left) by a coefficient called  $b_{i,j}$ . These coefficients are always assumed to be nonnegative (as before) and symmetric in  $i, j$  (that is,  $a_{i,j} = a_{j,i}$  and  $b_{i,j} = b_{j,i}$  for all  $i, j$ ). The corresponding to eq. (1.1) is then

$$(5.1) \quad \frac{d}{dt} c_i(t) = \frac{1}{2} \sum_{j=1}^{i-1} W_{j,i-j}(t) - \sum_{j=1}^{\infty} W_{i,j}(t), \quad i \in \mathbb{N}.$$

where

$$(5.2) \quad W_{i,j}(t) := a_{i,j} c_i(t) c_j(t) - b_{i,j} c_{i+j}(t) \quad i \in \mathbb{N}.$$

The system (1.1) is then a particular case of (5.1) obtained by choosing  $a_{i,j}$ ,  $b_{i,j}$  as

$$(5.3) \quad a_{i,j} = b_{i,j} = 0 \quad \text{when } \min\{i, j\} \geq 2,$$

$$(5.4) \quad a_{1,1} := 2a_1, \quad a_{i,1} = a_{1,i} = a_i \quad \text{for } i \geq 2,$$

$$(5.5) \quad b_{1,1} := 2b_2, \quad b_{i,1} = b_{1,i} = b_{i+1} \quad \text{for } i \geq 2.$$

The mathematical theory of this full system is much less complete than that of (1.1). Well-posedness of mass-conserving solutions has been studied in [2], and there are a number of works on asymptotic behaviour, for instance [12, 11, 8, 9], but it is still not fully understood. To start with, it is unclear whether equilibria of (5.1) are unique or not (when they exist). A common physical condition imposed on the coefficients  $a_{i,j}$ ,  $b_{i,j}$  which avoids this problem is that of *detailed balance*: we say it holds when there exists a sequence  $\{Q_i\}_{i \geq 1}$  of strictly positive numbers such that

$$(5.6) \quad a_{i,j}Q_iQ_j = b_{i,j}Q_{i+j} \quad \text{for any } i, j,$$

where we always further assume without loss of generality that  $Q_1 = 1$ . This is the analogue of (1.4), but in this case it needs to be imposed as a condition since numbers  $Q_i$  satisfying (5.6) cannot always be found (unlike in the Becker-Döring case). If we assume (5.6) then equilibria (5.1) exist and have the same form (1.5) as in the Becker-Döring case, and a similar phase transition in the long-time behaviour has been rigorously proved in some cases (see [12, 11, 8, 9] for more details). However, even with detailed balance the long-time behaviour is in general not understood except in particular cases. If clusters larger than a given size  $N$  do not react among themselves (that is, if  $a_{i,j} = b_{i,j} = 0$  whenever  $\min\{i, j\} > N$ ) the system is known as the *generalised Becker-Döring system*, and has been studied in [14, 8]. For coefficients  $a_{i,j}$  given by

$$(5.7) \quad a_{i,j} = i^\gamma j^\eta + i^\eta j^\gamma \quad \text{for any } i, j,$$

with  $\eta \leq 0 \leq \gamma$  and  $\gamma + \eta \leq 1$ , the asymptotic behaviour was identified in [9] and a constructive (though probably far from optimal) rate of convergence to equilibrium was given. Very little is known about the asymptotic behaviour for coefficients of the type (5.7) with  $\gamma, \eta > 0$  and  $\gamma + \eta \leq 1$ . In this case the size of  $a_{i,i}$  is larger than that of  $a_{i,1}$  and the system (5.1) may behave quite differently from (1.1).

The purpose of this section is to clarify whether any of the functional inequalities investigated in this paper can shed new light on the behaviour of solutions to (5.1). Assuming the detailed balance condition (5.6), along a solution  $\mathbf{c}(t) = \{c_i(t)\}_{i \geq 1}$  to (5.1) we have

$$(5.8) \quad \begin{aligned} \frac{d}{dt}H(\mathbf{c}(t)) &= -D_{\text{CF}}(\mathbf{c}(t)) \\ &:= -\frac{1}{2} \sum_{i,j=1}^{\infty} a_{i,j}Q_iQ_j \left( \frac{c_i c_j}{Q_i Q_j} - \frac{c_{i+j}}{Q_{i+j}} \right) \left( \log \frac{c_i c_j}{Q_i Q_j} - \log \frac{c_{i+j}}{Q_{i+j}} \right) \\ &\leq -\sum_{i=1}^{\infty} a_i Q_i \left( \frac{c_i c_1}{Q_i} - \frac{c_{i+1}}{Q_{i+1}} \right) \left( \log \frac{c_i c_1}{Q_i} - \log \frac{c_{i+1}}{Q_{i+1}} \right) = D(\mathbf{c}(t)) \leq 0 \end{aligned}$$

(see [9] for a rigorous proof) where  $a_i$  are defined by (5.4) for any  $i \geq 1$ . Hence the free energy is also a Lyapunov functional for (5.1), and it dissipates at a *faster* rate than for the Becker-Döring equations (since more types of reactions are allowed). As such, it is reasonable to think that the inequalities from Section 3 can be useful also in this case. This turns out to be true, and some improvements can be made on existing results. However, it also turns out that our results are not able to extend the range of possible

coefficients for which convergence to a particular subcritical equilibrium can be proved; we cannot give any new results for coefficients such as (5.7) with  $\gamma, \eta > 0$  and  $\gamma + \eta \leq 1$ .

One of the main obstacles in applying our results to equation (5.1) is that, unlike for the Becker-Döring equations, the moments of solutions to the general coagulation and fragmentation system are not known to be bounded. One can for example say the following about integer moments (this result can easily be extended to non-integer powers by interpolation, and was known from the early works in the topic [12, 11]). From this point onward we will assume that

$$(5.9) \quad a_{i,j} = i^\gamma j^\eta + i^\eta j^\gamma \quad \text{for } i, j \in \mathbb{N},$$

with  $\eta \leq \gamma$  and  $0 \leq \lambda := \gamma + \eta \leq 1$ .

**Lemma 5.1.** *Let  $k \in \mathbb{N}$  and let  $\mathbf{c} = \mathbf{c}(t) = \{c_i(t)\}_{i \in \mathbb{N}}$  be a solution with mass  $\varrho$  to the coagulation and fragmentation system (5.1) with coefficients satisfying (5.9). Then*

$$(5.10) \quad M_k(\mathbf{c}(t)) \leq \begin{cases} \left( M_k(\mathbf{c}(0)) + \frac{1-\lambda}{k-1} (2^k - 2) \varrho^{\frac{1-\gamma}{k-1}} t \right)^{\frac{k-1}{1-\lambda}} & \text{if } 0 < \lambda < 1 \\ M_k(\mathbf{c}(0)) \exp(2(2^k - 2) \varrho t) & \text{if } \lambda = 1 \end{cases}$$

where  $M_p(\mathbf{c}(t)) := \sum_{i=1}^{\infty} i^p c_i(t)$  for any  $p \geq 0$ ,  $t \geq 0$ .

*Proof.* We give a formal proof for completeness; a rigorous one can be obtained by standard approximation methods, and can be found in [2]. To simplify the notation and since  $\mathbf{c}(t)$  is fixed, we denote  $M_j(t) = M_j(\mathbf{c}(t))$  for any  $j \geq 1$ ,  $t \geq 0$ . One can check the following weak formula for the integral of the right hand side of (5.1) against a test sequence  $\{\phi(i)\}_i$ :

$$\sum_{i=1}^{\infty} \phi(i) \left( \frac{1}{2} \sum_{j=1}^{i-1} W_{i-j,j} - \sum_{j=1}^{\infty} W_{i,j} \right) = \frac{1}{2} \sum_{i,j} (\phi(i+j) - \phi(i) - \phi(j)) W_{i,j}.$$

Applying this to  $\phi(i) := i^k$ , neglecting the negative contribution of the fragmentation terms and using the binomial formula one obtains

$$\frac{d}{dt} M_k(t) \leq \sum_{l=1}^{k-1} \binom{k}{l} M_{l+\gamma}(t) M_{k-l+\eta}(t) \quad \forall t \geq 0.$$

Next, we use the interpolation

$$M_\delta(t) \leq M_1^{\frac{k-\delta}{k-1}}(t) M_k^{\frac{\delta-1}{k-1}}(t)$$

where  $1 < \delta < k$ , to find that

$$M_{l+\gamma}(t) M_{k-l+\eta}(t) \leq M_1(t)^{\frac{k-\lambda}{k-1}} M_k(t)^{\frac{k+\lambda-2}{k-1}}.$$

Thus,

$$\frac{d}{dt} M_k(t) \leq (2^k - 2) \varrho^{\frac{k-\lambda}{k-1}} M_k^{\frac{k+\lambda-2}{k-1}}(t) \quad \forall t \geq 0$$

and the result follows from this differential inequality.  $\square$

With the above at hand, we are now able to use the theory developed in the previous sections for the Becker-Döring equations in order to conclude a rate of convergence to equilibrium in the general setting of coagulation and fragmentation equations. Our main theorem is the following:

**Theorem 5.2** (Asymptotic behaviour of the coagulation-fragmentation system). *Let  $\{a_{i,j}\}_{i,j \in \mathbb{N}}, \{b_{i,j}\}_{i,j \in \mathbb{N}}$  be the coagulation and fragmentation coefficients for equation (5.1), and assume that the detailed balance condition (5.6) holds. Assume that*

$$(5.11) \quad a_{i,j} = i^\gamma + j^\gamma,$$

*for some  $0 \leq \gamma < 1$  and that  $\{Q_i\}_{i \in \mathbb{N}}$  satisfies Hypothesis 2. Assume in addition that  $M_k(\mathbf{c}(0)) < +\infty$  for some  $k \in \mathbb{N}$ ,  $k > 1$ . Then*

$$(5.12) \quad H(\mathbf{c}(t)|\mathcal{Q}) \leq \frac{1}{(C_1 + C_2 \log t)^{\frac{k-1}{1-\gamma}}}$$

*where  $C_1, C_2 > 0$  are constants depending only on  $H(\mathbf{c}(0)|\mathcal{Q}), z_s, \varrho, \{\alpha_i\}_{i \in \mathbb{N}}, k, \gamma$  and  $M_k(\mathbf{c}(0))$ .*

*Proof.* Assume for the moment that  $a_{i,j}$  is of the form (5.7), in order to see why the proof only works for coefficients of the form (5.11).

Fix  $\delta > 0$  such that  $0 < \delta < \bar{z} < z_s - \delta$ . We use the observation (5.8) that  $D_{\text{CF}}(\mathbf{c}(t)) \geq D(\mathbf{c}(t))$  at all times  $t \geq 0$  (defining  $\{a_i\}_{i \in \mathbb{N}}$  by (5.4)). Using Theorem 1.1 (actually, its more detailed forms in equation (3.18) and Proposition 3.8) we obtain the following:

$$\begin{aligned} \frac{d}{dt} H(\mathbf{c}(t)|\mathcal{Q}) &= -D_{\text{CF}}(\mathbf{c}(t)) \leq -D(\mathbf{c}(t)) \\ &\leq \begin{cases} -CM_k(\mathbf{c}(t))^{\frac{\gamma-1}{k-1}} H(\mathbf{c}(t)|\mathcal{Q})^{\frac{k-\gamma}{k-1}} & \text{if } \delta < c_1(t) < z_s - \delta \\ -CM_k(\mathbf{c}(t))^{\frac{\gamma-1}{k-1}} & \text{if } c_1(t) < \delta \text{ or } c_1(t) \geq z_s - \delta. \end{cases} \\ &\leq -C_0 M_k(\mathbf{c}(t))^{\frac{\gamma-1}{k-1}} H(\mathbf{c}(t)|\mathcal{Q})^{\frac{k-\gamma}{k-1}} \end{aligned}$$

for some constant  $C_0 > 0$  that depends also on  $H(\mathbf{c}(0)|\mathcal{Q})$ . Using Lemma 5.1 this implies

$$\frac{d}{dt} H(\mathbf{c}(t)|\mathcal{Q}) \leq -\frac{C_0}{\left(M_k(\mathbf{c}(0)) + \frac{1-\lambda}{k-1}(2^k - 2)\varrho^{\frac{k-\lambda}{k-1}}t\right)^{\frac{1-\gamma}{1-\lambda}}} H(\mathbf{c}(t)|\mathcal{Q})^{\frac{k-\gamma}{k-1}} \quad t \geq 0.$$

This implies decay of  $H(\mathbf{c}(t))$  only when  $\lambda = \gamma$ , that is, when  $\eta = 0$  (since  $\lambda = \gamma + \eta$ ). Solving the differential inequality yields the result.  $\square$

*Remark 5.3.* The same decay rate was obtained in [9] by means of the particular case of inequality (1.21) for  $k = 2 - \gamma$ . Here we obtain slightly different decay rates by assuming higher moments of the initial data  $\mathbf{c}(0)$  are finite, but the method does not seem to give a better decay than a power of  $\log t$  in any case.

## 6. FINAL REMARKS.

In this final section we gather a few remarks and open problems associated to this paper:

**The assumption that  $c_1(t)$  is in the 'good' region** given by (1.18) is not so far-fetched. Indeed, as long as we know that the energy decreases in some given way, we can apply the following Csiszár-Kullback inequality, (1.15),

$$\sum_{i=1}^{\infty} |c_i(t) - \mathcal{Q}_i| \leq \sqrt{2\varrho H(\mathbf{c}(t)|\mathcal{Q})}$$

and obtain that if  $H(\mathbf{c}(t_0)|\mathcal{Q})$  is small enough then for any  $t > t_0$

$$\bar{z} - H(\mathbf{c}(t_0)|\mathcal{Q}) \leq c_1(t) \leq \bar{z} + H(\mathbf{c}(t_0)|\mathcal{Q}).$$

**The hypothesis on the form of  $\{Q_i\}_{i \in \mathbb{N}}$** , stated in Hypothesis 2, was used explicitly in our work in order to obtain very *quantitative* estimations on the constants appearing in our theorems. As one can see from the proofs, this hypothesis can be relaxed - but the price one must pay is losing that quantified estimate.

**Considering the rate of convergence to equilibrium when  $\gamma < 1$** , one may wonder if, at least for initial data close enough to equilibrium, the rate of convergence to equilibrium in the case when  $a_i = i^\gamma$  with  $\gamma < 1$  can be improved to an exponential. Recently, Murray and Pego investigated this rate of convergence only to conclude an algebraic form of decay (see [18]). It would be interesting to verify the optimality of this result by determining whether the linearised operator for the equations admits a spectral gap in  $\ell^1$  spaces with polynomial weights (in  $\ell^1$  spaces with exponential weights, the answer is positive and an estimate of the spectral gap can be found in [10]). The authors believe that no such spectral gap exists for  $0 \leq \gamma < 1$ , i.e. the algebraic rate of convergence is optimal even for close to equilibrium initial data.

**Consider the general coagulation and fragmentation equations** presented in Section 5. It seems to the authors that the method used is too crude and is far from optimal. We suspect that the inequality obtained in Theorem 5.2 can be improved to deal with the case

$$a_{i,j} = i^\gamma j^\eta + i^\eta j^\gamma$$

and the resulting convergence rate will depend on  $\lambda = \gamma + \eta$ .

The authors are curious to see if some of the aforementioned problems can be addressed with the help of the presented work, and are eager to see possible new functional inequalities arising in connection to the Becker-Döring equations, or more general coagulation and fragmentations models.

## APPENDIX A. ADDITIONAL COMPUTATIONS FOR THE THEORY OF THE DISCRETE LOG-SOBOLEV WITH WEIGHTS INEQUALITY

We have collected here technical Lemmas from Subsection 2 that we felt would have encumbered the flow of it.

**Lemma A.1.** *For any sequence  $\mathbf{f}$ , we have*

$$\mathbf{Ent}_\mu(\mathbf{f}^2) \leq \mathcal{L}(\mathbf{f}) \leq \mathbf{Ent}_\mu(\mathbf{f}^2) + 2 \sum_{i=1}^{\infty} \mu_i f_i^2.$$

*Proof.* From the definition of  $\mathcal{L}$  the inequality

$$\mathbf{Ent}_\mu(\mathbf{f}^2) \leq \mathcal{L}(\mathbf{f})$$

is trivial. We thus consider the right hand side inequality. For a given sequence  $\mathbf{f}$  and any  $\alpha \in \mathbb{R}$  we define

$$\begin{aligned} G_\alpha(t) &= \sum_{i=1}^{\infty} \mu_i (tf_i + \alpha)^2 \log \left( \frac{(tf_i + \alpha)^2}{\sum_{i=1}^{\infty} \mu_i (tf_i + \alpha)^2} \right) \\ &= 2 \sum_{i=1}^{\infty} \mu_i (tf_i + \alpha)^2 \log |tf_i + \alpha| - \left( \sum_{i=1}^{\infty} \mu_i (tf_i + \alpha)^2 \right) \log \left( \sum_{i=1}^{\infty} \mu_i (tf_i + \alpha)^2 \right), \end{aligned}$$

and notice that

$$G_0(t) = t^2 \mathbf{Ent}_\mu(\mathbf{f}^2).$$

Next, we define  $g(t) = G_0(t) + 2t^2 \sum_{i=1}^\infty \mu_i f_i^2$  and notice that the inequality we want to prove is equivalent to

$$G_\alpha(1) \leq g(1).$$

for any  $\alpha \in \mathbb{R}$ . Clearly  $G_\alpha(t) \leq g(t)$  when  $t = 0$ . Differentiating  $G$  we find that

$$\begin{aligned} G'_\alpha(t) &= 4 \sum_{i=1}^\infty \mu_i f_i |tf_i + \alpha| \log(tf_i + \alpha) + 2 \sum_{i=1}^\infty \mu_i f_i (tf_i + \alpha) \\ &\quad - 2 \left( \sum_{i=1}^\infty \mu_i f_i (tf_i + \alpha) \right) \log \left( \sum_{i=1}^\infty \mu_i (tf_i + \alpha)^2 \right) - 2 \sum_{i=1}^\infty \mu_i f_i (tf_i + \alpha) \\ &= 4 \sum_{i=1}^\infty \mu_i f_i (tf_i + \alpha) \log |tf_i + \alpha| - 2 \left( \sum_{i=1}^\infty \mu_i f_i (tf_i + \alpha) \right) \log \left( \sum_{i=1}^\infty \mu_i (tf_i + \alpha)^2 \right) \end{aligned}$$

which satisfies  $G'_\alpha(0) = 0$  for any  $\mathbf{f}$  and  $\alpha$ , implying that  $G'_\alpha(0) = g'(0) = 0$ . As  $G$  is defined for any  $t \in [0, 1]$  we see that it is enough to show that when defined,

$$G''_\alpha(t) \leq g''(t)$$

for any  $\alpha$ . Indeed,

$$\begin{aligned} G''_\alpha(t) &= 4 \sum_{i=1}^\infty \mu_i f_i^2 \log |tf_i + \alpha| + 4 \sum_{i=1}^\infty \mu_i f_i^2 - 2 \sum_{i=1}^\infty \mu_i f_i^2 \log \left( \sum_{i=1}^\infty \mu_i (tf_i + \alpha)^2 \right) \\ &\quad - 4 \frac{(\sum_{i=1}^\infty \mu_i f_i (tf_i + \alpha))^2}{\sum_{i=1}^\infty \mu_i (tf_i + \alpha)^2} \\ &= 2 \sum_{i=1}^\infty \mu_i f_i^2 \log \left( \frac{(tf_i + \alpha)^2}{\sum_{i=1}^\infty \mu_i (tf_i + \alpha)^2} \right) + 4 \sum_{i=1}^\infty \mu_i f_i^2 - 4 \frac{(\sum_{i=1}^\infty \mu_i f_i (tf_i + \alpha))^2}{\sum_{i=1}^\infty \mu_i (tf_i + \alpha)^2} \end{aligned}$$

As

$$\mathbf{Ent}_\mu(\mathbf{f}^2) = \sup \left\{ \sum_{i=1}^\infty \mu_i f_i^2 \log h_i ; \sum_{i=1}^\infty \mu_i h_i = 1 \right\}$$

we see that by choosing  $h_i = \frac{(tf_i + \alpha)^2}{\sum_{i=1}^\infty \mu_i (tf_i + \alpha)^2}$

$$G''_\alpha(t) \leq 2 \mathbf{Ent}_\mu(\mathbf{f}^2) + 4 \sum_{i=1}^\infty \mu_i f_i^2 = g''(t),$$

completing the proof. □

**Lemma A.2.** For all  $\mathbf{f} \in L_\Phi$  we have that

$$(A.1) \quad \|\mathbf{f}\|_{L_\mu^1} \leq \|\mathbf{f}\|_{L_\mu^2} \leq \sqrt{\frac{3}{2}} \|\mathbf{f}\|_{L_\Phi}.$$

*Proof.* The inequality

$$\|\mathbf{f}\|_{L_\mu^1} \leq \|\mathbf{f}\|_{L_\mu^2}$$

is immediate as  $\mu$  is a probability measure. To show the last inequality we may assume that  $\|\mathbf{f}\|_{L_\Phi} = 1$ . Due to Fatou's Lemma we know that if  $k_n \xrightarrow{n \rightarrow \infty} k > 0$  then

$$\sum_{i=1}^{\infty} \mu_i \Phi\left(\frac{|f_i|}{k}\right) \leq \liminf_{n \rightarrow \infty} \sum_{i=1}^{\infty} \mu_i \Phi\left(\frac{|f_i|}{k_n}\right),$$

implying that if  $\|f\|_{L_\Phi} > 0$  then

$$\sum_{i=1}^{\infty} \mu_i \Phi\left(\frac{|f_i|}{\|f\|_{L_\Phi}}\right) \leq 1.$$

In our case, since  $\Psi(x)$  is convex we find that

$$1 \geq \sum_{i=1}^{\infty} \mu_i \Phi(f_i) = \sum_{i=1}^{\infty} \mu_i \Psi(f_i^2) \geq \Psi\left(\sum_{i=1}^{\infty} \mu_i f_i^2\right) = \Psi\left(\|\mathbf{f}\|_{L_\mu}^2\right).$$

As  $\Psi$  is increasing and  $\Psi(1.5) > 1$  we conclude that

$$\|\mathbf{f}\|_{L_\mu}^2 < \frac{3}{2},$$

yielding the desired result. □

**Lemma A.3.** *Let  $\mathbf{f} \in L_\Phi$ . Then*

$$(A.2) \quad \|\mathbf{f} - \langle \mathbf{f} \rangle\|_{L_\mu}^2 = \frac{1}{2} \lim_{|a| \rightarrow \infty} \mathbf{Ent}_\mu((\mathbf{f} + a)^2)$$

*Proof.* We start by noticing that

$$\mathbf{Ent}_\mu((\mathbf{f} + a)^2) = \sum_{i=1}^{\infty} \mu_i (f_i^2 + 2af_i + a^2) \log\left(\frac{(1 + \frac{f_i}{a})^2}{\sum_{i=1}^{\infty} \mu_i (1 + \frac{f_i}{a})^2}\right),$$

and continue by assuming that  $f_i$  is uniformly bounded, from which the result will follow with an application of an appropriate convergence theorem. There exists  $a_0$  such that if  $|a| > |a_0|$  we have that  $|\frac{f_i}{a}| < \frac{1}{2}$  uniformly in  $i$ . As on  $[-\frac{1}{2}, \frac{1}{2}]$  we have that there exists  $C > 0$  such that

$$\left|\log(1+x) - x + \frac{x^2}{2}\right| \leq Cx^3.$$

we conclude that

$$\log\left(1 + 2\frac{f_i}{a} + \frac{f_i^2}{a^2}\right) = \left(2\frac{f_i}{a} + \frac{f_i^2}{a^2}\right) - 2\frac{f_i^2}{a^2} + \frac{E_{1,i}}{a^3} = 2\frac{f_i}{a} - \frac{f_i^2}{a^2} + \frac{E_{1,i}}{a^3}$$

and

$$\log\left(1 + 2\frac{\langle \mathbf{f} \rangle}{a} + \frac{\|\mathbf{f}\|_{L_\mu}^2}{a^2}\right) = 2\frac{\langle \mathbf{f} \rangle}{a} + \frac{\|\mathbf{f}\|_{L_\mu}^2}{a^2} - 2\frac{\langle \mathbf{f} \rangle^2}{a^2} + \frac{E_{2,i}}{a^3},$$

where  $E_{1,i}, E_{2,i}$  are uniformly bounded in  $i$ . This implies that

$$\begin{aligned} \mathbf{Ent}_\mu((\mathbf{f} + a)^2) &= \sum_{i=1}^{\infty} \mu_i (f_i^2 + 2af_i + a^2) \left(2\frac{f_i}{a} - 2\frac{\langle \mathbf{f} \rangle}{a} - \frac{f_i^2}{a^2} - \frac{\|\mathbf{f}\|_{L_\mu}^2}{a^2} + 2\frac{\langle \mathbf{f} \rangle^2}{a^2}\right) \\ &\quad + \frac{1}{a} \sum_{i=1}^{\infty} \mu_i \left(1 + 2\frac{f_i}{a} + \frac{f_i^2}{a^2}\right) (E_{1,i} - E_{2,i}). \end{aligned}$$



The last term clearly goes to zero as  $|a|$  goes to infinity, so we are only left to deal with the first expression.

$$\begin{aligned} \sum_{i=1}^{\infty} \mu_i (f_i^2 + 2af_i + a^2) \left( 2\frac{f_i}{a} - 2\frac{\langle \mathbf{f} \rangle}{a} - \frac{f_i^2}{a^2} - \frac{\|\mathbf{f}\|_{L_\mu^2}^2}{a^2} + 2\frac{\langle \mathbf{f} \rangle^2}{a^2} \right) &= 4\|\mathbf{f}\|_{L_\mu^2}^2 - 4\langle \mathbf{f} \rangle^2 \\ &\quad + 2a\langle \mathbf{f} \rangle - 2a\langle \mathbf{f} \rangle - \|\mathbf{f}\|_{L_\mu^2}^2 - \|\mathbf{f}\|_{L_\mu^2}^2 + 2\langle \mathbf{f} \rangle^2 + \frac{E_3}{a} \\ &= 2\left(\|\mathbf{f}\|_{L_\mu^2}^2 - \langle \mathbf{f} \rangle^2\right) + \frac{E_3}{a}. \end{aligned}$$

This completes the proof as  $\|\mathbf{f} - \langle \mathbf{f} \rangle\|_{L_\mu^2}^2 = \|\mathbf{f}\|_{L_\mu^2}^2 - \langle \mathbf{f} \rangle^2$ .  $\square$

**Lemma A.4.** *Let  $\mathbf{f}$  be a sequence such that  $f_m = 0$  for some  $m \in \mathbb{N}$ . Denote by  $\mathbf{f}^{(0)} = \mathbf{f} \mathbb{1}_{i < m}$  and  $\mathbf{f}^{(1)} = \mathbf{f} \mathbb{1}_{i > m}$ . Then*

$$\begin{aligned} (A.3) \quad \|\langle \mathbf{f}^{(0)} \rangle\|_{L_\Phi} &\leq |\langle \mathbf{f}^{(0)} \rangle| \leq \|\mathbf{f}^{(0)}\|_{L_\mu^2} \sqrt{\sum_{i=1}^{m-1} \mu_i} \\ \|\langle \mathbf{f}^{(1)} \rangle\|_{L_\Phi} &\leq |\langle \mathbf{f}^{(1)} \rangle| \leq \|\mathbf{f}^{(1)}\|_{L_\mu^2} \sqrt{\sum_{i=m+1}^{\infty} \mu_i} \end{aligned}$$

*Proof.* We start by noticing that for any constant sequence  $\mathbf{f} = \alpha$  one have

$$\begin{aligned} \|\alpha\|_{L_\Phi} &= \inf_{k>0} \left\{ \sum_{i=1}^{\infty} \mu_i \Phi\left(\frac{|\alpha|}{k}\right) \leq 1 \right\} = \inf_{k>0} \left\{ \Phi\left(\frac{|\alpha|}{k}\right) \leq 1 \right\} \\ &= \frac{|\alpha|}{\Phi^{-1}(1)} \leq |\alpha|, \end{aligned}$$

as long as  $\Phi(1) < 1$  which is valid in our case. Next we notice that

$$|\langle \mathbf{f}^{(0)} \rangle| \leq \sum_{i=1}^{m-1} \mu_i |f_i| \leq \sqrt{\sum_{i=1}^{m-1} \mu_i f_i^2} \sqrt{\sum_{i=1}^{m-1} \mu_i} = \|\mathbf{f}^{(0)}\|_{L_\mu^2} \sqrt{\sum_{i=1}^{m-1} \mu_i}.$$

This yields the first inequality and similar arguments yield the second inequality.  $\square$

*Remark A.5.* As was shown in the proof of Lemma A.4 one can actually improve the bounds in (A.3) by a factor of  $\Psi^{-1}(1)$ .

**Lemma A.6.** *For any  $t \geq \frac{3}{2}$  one has that*

$$(A.4) \quad \frac{1}{3} \frac{t}{\log t} \leq \Psi^{-1}(t) \leq 2 \frac{t}{\log t}.$$

*Proof.* We start by noticing that

$$\begin{aligned} \Psi\left(\frac{1}{3} \frac{t}{\log t}\right) &= \frac{1}{3} \frac{t}{\log t} \log\left(1 + \frac{1}{3} \frac{t}{\log t}\right) \leq \frac{1}{3} \frac{t}{\log t} \log\left(1 + \frac{t}{\log\left(\frac{27}{8}\right)}\right) \\ &\leq \frac{1}{3} \frac{t}{\log t} \log(1+t). \end{aligned}$$

Thus, one notices that if

$$1+t \leq t^3$$

when  $t \geq \frac{3}{2}$ , we have that  $\Psi\left(\frac{1}{3}\frac{t}{\log t}\right) \leq t$ , yielding the left hand side of (A.4). This is indeed the case as  $g(t) = t^3 - t - 1$  is increasing on  $\left[\frac{1}{\sqrt{3}}, \infty\right)$  and  $g\left(\frac{3}{2}\right) > 0$ . For the converse we notice that

$$\Psi\left(2\frac{t}{\log t}\right) = 2\frac{t}{\log t} \log\left(1 + 2\frac{t}{\log t}\right) \geq t$$

if and only if

$$1 + 2\frac{t}{\log t} \geq \sqrt{t}.$$

Considering the function  $g(x) = \frac{x}{\log x}$  for  $x > 1$  we see that it obtains a minimum at  $x = e$ . Thus, for any  $x > 1$   $g(x) \geq e > 1$ . We conclude that for  $t > \frac{3}{2}$

$$2\frac{t}{\log t} = \sqrt{t}g(\sqrt{t}) \geq \sqrt{t},$$

showing the desired result.  $\square$

## APPENDIX B. ADDITIONAL USEFUL COMPUTATIONS

**Lemma B.1.** *For a given coagulation and detailed balance coefficients,  $\{a_i\}_{i \in \mathbb{N}}$ ,  $\{Q_i\}_{i \in \mathbb{N}}$ , and a given positive sequence  $\mathbf{c}$  with finite mass  $\varrho$ , we have that for any  $z > 0$*

$$H(\mathbf{c}|\mathcal{Q}) \leq H(\mathbf{c}|\mathcal{Q}_z),$$

where  $\mathcal{Q} = \mathcal{Q}_{\bar{z}}$ .

*Proof.* We have that

$$H(\mathbf{c}|\mathcal{Q}_z) = \sum_{i=1}^{\infty} c_i \left( \log\left(\frac{c_1}{Q_i z^i}\right) - 1 \right) + \sum_{i=1}^{\infty} Q_i z^i$$

implying that

$$H(\mathbf{c}|\mathcal{Q}_{z_1}) - H(\mathbf{c}|\mathcal{Q}_{z_2}) = \sum_{i=1}^{\infty} i c_i \log\left(\frac{z_2}{z_1}\right) + \sum_{i=1}^{\infty} Q_i (z_1^i - z_2^i).$$

In particular, if  $z_2 = \bar{z}$  we have that for any  $z > 0$

$$\begin{aligned} H(\mathbf{c}|\mathcal{Q}_z) &= H(\mathbf{c}|\mathcal{Q}) + \varrho \log\left(\frac{\bar{z}}{z}\right) + \sum_{i=1}^{\infty} Q_i (z^i - \bar{z}^i) \\ &= H(\mathbf{c}|\mathcal{Q}) + \sum_{i=1}^{\infty} i Q_i \bar{z}^i \log\left(\frac{\bar{z}}{z}\right) + \sum_{i=1}^{\infty} Q_i z^i \left(1 - \left(\frac{\bar{z}}{z}\right)^i\right) \\ &= H(\mathbf{c}|\mathcal{Q}) + \sum_{i=1}^{\infty} Q_i z^i \left( \left(\frac{\bar{z}}{z}\right)^i \log\left(\left(\frac{\bar{z}}{z}\right)^i\right) - \left(\frac{\bar{z}}{z}\right)^i + 1 \right) \\ &= H(\mathbf{c}|\mathcal{Q}) + \sum_{i=1}^{\infty} Q_i z^i \Delta\left(\frac{(\mathcal{Q}_z)_i}{Q_i}\right), \end{aligned}$$

where  $\Delta(x) = x \log x - x + 1 > 0$  when  $x > 0$ . This completes the proof.  $\square$

**Lemma B.2.** Let  $\{Q_i\}_{i \in \mathbb{N}}$  be a non-negative sequence such that  $\lim_{i \rightarrow \infty} \frac{Q_{i+1}}{Q_i} = \frac{1}{r}$  for some  $r > 0$ . Assume that  $0 < x < r_1 < r$ . Then

$$\sum_{i=i_0+1}^{\infty} i Q_i x^{i-1} \leq C Q_{i_0} x^{i_0},$$

where  $C$  is a constant depending only on  $\{Q_i\}_{i \in \mathbb{N}}$  and  $r_1$ .

*Proof.* Define  $\beta_i = \frac{Q_{i+1}}{Q_i}$ . We have that  $\lim_{i \rightarrow \infty} \beta_i = \frac{1}{r}$ , and as such we can find  $l \in \mathbb{N}$  such that for all  $i > l$

$$\Lambda_1 = \sup_{i > l} \beta_i < \frac{1}{r_1}.$$

Denote  $\Lambda_2 = \sup_{i \leq l} \beta_i$ . As for any  $i > i_0$

$$Q_i = \left( \prod_{j=i_0}^{i-1} \beta_j \right) Q_{i_0}$$

we see that

$$\begin{aligned} \sum_{i=i_0+1}^{\infty} i Q_i x^{i-1} &= Q_{i_0} x^{i_0} \sum_{i=i_0+1}^{\infty} i \left( \prod_{j=i_0}^{i-1} \beta_j \right) x^{i-i_0-1} \leq Q_{i_0} x^{i_0} \left( \Lambda_2 \sum_{j=0}^{l-i_0} i (\Lambda_2 r_1)^j + \Lambda_1 \sum_{j=l+1-i_0}^{\infty} i (\Lambda_1 r_1)^j \right) \\ &\leq Q_{i_0} x^{i_0} \left( \Lambda_2 \sum_{j=0}^l j (\Lambda_2 r_1)^j + \Lambda_1 \sum_{j=0}^{\infty} j (\Lambda_1 r_1)^j \right) \end{aligned}$$

completing the proof as  $l, \Lambda_1$  and  $\Lambda_2$  depend solely on  $\{Q_i\}_{i \in \mathbb{N}}$  □

**Lemma B.3.** Let  $\varepsilon > 0$  and  $\gamma > 0$ . Denote by

$$B_{\varepsilon, \gamma} = \sum_{i=1}^{\infty} i^{\gamma} e^{-\varepsilon i}.$$

Then  $\varepsilon^{1+\gamma} B_{\varepsilon, \gamma}$  is of order 1 when  $\varepsilon$  goes to zero.

*Proof.* We start by noticing that the function  $g_{\varepsilon, \gamma}(x) = x^{\gamma} e^{-\varepsilon x}$  is increasing in  $[0, \frac{\gamma}{\varepsilon}]$  and decreasing in  $[\frac{\gamma}{\varepsilon}, \infty)$ . As such

$$\begin{aligned} B_{\varepsilon, \gamma} &\geq \sum_{i=\lceil \frac{\gamma}{\varepsilon} \rceil + 1}^{\infty} i^{\gamma} e^{-\varepsilon i} \geq \int_{\lceil \frac{\gamma}{\varepsilon} \rceil + 1}^{\infty} x^{\gamma} e^{-\varepsilon x} dx \\ &= \varepsilon^{-(1+\gamma)} \int_{\varepsilon(\lceil \frac{\gamma}{\varepsilon} \rceil + 1)}^{\infty} y^{\gamma} e^{-y} dy \geq \varepsilon^{-(1+\gamma)} \int_{\varepsilon}^{\infty} y^{\gamma} e^{-y} dy, \end{aligned}$$

showing the lower bound. For the upper bound we notice that

$$B_{\varepsilon, \gamma} \leq \sup_{x \geq 0} g_{\varepsilon, \gamma}(x) \sum_{i=1}^{\infty} e^{-\frac{\varepsilon}{2} i} = \left( \frac{2\gamma}{\varepsilon} \right)^{\gamma} e^{-\gamma} \frac{e^{-\frac{\varepsilon}{2}}}{1 - e^{-\frac{\varepsilon}{2}}}$$

which completes the proof since  $\sup_{\varepsilon > 0} \frac{\varepsilon e^{-\frac{\varepsilon}{2}}}{1 - e^{-\frac{\varepsilon}{2}}} < +\infty$ . □

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